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PATH-SPACES ADMITTING COLLINEATIONS

By D. D. KOSAMBI (*Bombay*)

[Received 4 April 1951]

THIS note is concerned with path-equations

$$\ddot{x}^i+\alpha^i(x,\dot{x}) = 0, \quad \dot{x}^i = dx^i/dt, \quad \ddot{x}^i = d\dot{x}^i/dt \quad (i = 1, 2,..., n) \tag{1}$$

which are preserved under a continuous group of transformations (other than the identity) of the base-space $x^1...x^n$ into itself. It has been shown in a previous communication (**1**) that this group, or rather the Lie algebra of its generators, is precisely defined by the vector solutions $u^i(x)$ of the equations of variation of the paths

$$\ddot{u}^i+\alpha^i_{;r}\dot{u}^r+\alpha^i_{,r}u^r = 0, \tag{2}$$

where the notation is

$$\dot{u}^i = du^i/dt, \text{ etc.}, \qquad d/dt = \partial/\partial t+\dot{x}^r\,\partial/\partial x^r-\alpha^r\,\partial/\partial\dot{x}^r,$$

$$F^{\cdot}_{;r} = \partial F^{\cdot}_{\cdot}/\partial\dot{x}^r, \qquad F^{\cdot}_{\cdot,r} = \partial F^{\cdot}_{\cdot}/\partial x^r,$$

and the tensor summation convention is used for an index repeated in subscript and superscript. When a function $f(x,\dot{x})$ exists such that (1) represent the extremals of a regular principle $\delta\int f\,dt = 0$, we say that the path-space 'admits a metric'. A necessary and sufficient condition therefore is that the equations of variation be self-adjoint (**2**). Inasmuch as no metric need exist, the usual group-theoretic treatment of a differential manifold is not applicable, for it implies the existence of a Riemann metric. However, one may still proceed by formal power series, neglecting topological considerations (**3**).

The system of partial differential equations (2) for the generating vectors $u^i(x)$ may be rewritten in the form

$$u^i_{,j,k}\dot{x}^j\dot{x}^k-u^i_{,r}\alpha^r+\alpha^i_{;r}u^r_{,j}\dot{x}^j+\alpha^i_{,r}u^r = 0. \tag{3}$$

With the solution $u^i(x)$ is associated the Lie operator L, defined for an ordinary function $f(x)$ by $Lf = u^rf_{,r}$; for $f(x,\dot{x})$, its first extension, $Lf = \dot{u}^rf_{;r}+u^rf_{,r}$. For tensors of weight p, this becomes

$$LT^{i\cdot}_{j\cdot} = \dot{u}^rT^{i\cdot}_{j\cdot;r}+u^rT^{i\cdot}_{j\cdot,r}+u^r_{,j}T^{i\cdot}_{r\cdot}+...-u^i_{,r}T^{r\cdot}_{j\cdot}-...+pu^r_{,r}T^{i\cdot}_{j\cdot}. \tag{4}$$

For general geometric objects, i.e. functions of x, $\dot{x}$ possessing a transformation law, whether tensors or not, L may be defined as the difference between the (infinitesimal) changes made in the geometric object by first regarding the (infinitesimal) transformation $\bar{x}^i = x^i+u^i\,\delta\tau$

as a change of variables and then as a transformation of coordinates. Thus (2) and (3) are merely $L\alpha^i = 0$. For any geometric object, the formal power series expansion $\bar{F} = (\exp \tau L)F$ holds under the finite transformation of the group.

The path-spaces (1) above include, as special cases, those with symmetric affine connexion, where $\alpha^i = \Gamma^i_{jk}\dot{x}^j\dot{x}^k$. These in turn include Riemann spaces, the paths being the geodesics and Γ^i_{jk} Christoffel symbols, with a metric tensor g_{ij}. In general, we have fibre-spaces, $c\dot{x}^i$ being the fibre-bundle attached to each point (x) of the base-space. Our methods extend immediately to differential equations of higher order explicitly soluble for the highest derivative, and also to partial differential equations of similar type. The extension to $\alpha^i(x, \dot{x}, t)$ causes no difficulty. All these equations may be taken in the abstract, by assigning a suitable meaning to the summation convention (after Michal, say, replacing summation by an integration), and to differentiation.

The operator L, besides being a tensorial Lie derivative, has the properties

$$L\dot{x}^i = 0, \qquad L_u\, v^i = (u, v) = u^r v^i_{\ ,r} - v^r u^i_{\ ,r} \quad \text{(the Poisson bracket);}$$

the equations of variation even for $u^i(x, \dot{x})$ form the *ring* of all operators that commute with d/dt. We shall use the

LEMMA. *All differential operators of the path-space commute with L; also, the integrability conditions of* (3) *say merely that L preserves all differential invariants of the space.*

It is seen at once that L permutes with $\partial/\partial\dot{x}^i$, and its commuting also with d/dt suffices to prove the statement for all operators of the space. For differential invariants, it is fairly obvious that the transformation group that preserves the paths also preserves the differential invariants calculated from the paths. That the compatibility conditions do not imply anything more need not be proved in detail here, for it is a much more general theorem, holding for affine and projective collineations, as may be seen (though never explicitly stated) by a survey of the literature, say in the summary of Yano (**4**). The failure to recognize the meaning of the compatibility conditions in terms of L usually leads to laborious but unnecessary calculations, which we shall avoid.

For the particular path-equations (1), the actual basis of differential operators is, taking connexion coefficients $\gamma^i_j = \frac{1}{2}\alpha^i_{\ ;j}$

$$\frac{\partial}{\partial\dot{x}^i}, \qquad DT^i_j = \frac{d}{dt}T^i_j + \tfrac{1}{2}\alpha^i_{\ ;r}T^r_j - \tfrac{1}{2}\alpha^r_{\ ;j}T^i_r - p\,.\,\tfrac{1}{2}\alpha^r_{\ ;r}T^i_j \tag{5}$$

and the covariant derivative $(\partial/\partial\dot{x}^i)D-D(\partial/\partial\dot{x}^i)$. The basis for differential invariants is, neglecting $\dot{x}^i$,

$$\epsilon^i = \alpha^i - \tfrac{1}{2}\dot{x}^r\alpha^i{}_{;r}, \qquad \tfrac{1}{2}\alpha^i{}_{;j;k;l}, \tag{6}$$

and
$$P^i_j = -\alpha^i{}_{,j} + \frac{d}{dt}\alpha^i{}_{;j} + \tfrac{1}{4}\alpha^i{}_{;r}\alpha^r{}_{;j}.$$

1. If a transformation group does not affect all n dimensions of the space, the geometry is naturally reduced to a lower number of dimensions. Hence we may regard such groups as of secondary importance. With this restriction, we have

THEOREM 1.1. *The paths admit the translation group T_n if and only if the equations are $\ddot{x}^i+\alpha^i(\dot{x}) = 0$. They admit a subgroup of the homogeneous linear group if and only if they are of the form*

$$\ddot{x}^i+\dot{x}^iH(x,\dot{x})+x^iJ(x,\dot{x}) = 0,$$

where H and J are absolute invariants of the (first extension of the) subgroup in question.

Proof. The operator of a translation along the jth coordinate direction is $\partial/\partial x^j$, which means that $\alpha^i{}_{,j} = 0$, x^j not appearing explicitly in the α^i. The generators of the homogeneous linear group are all of form $x^k\partial/\partial x^j$, which means that α^k must be a linear combination of x^k and $\dot{x}^k$. Finally, when $u^i(x)$ is linear, its second partial derivatives vanish so that (3) merely shows α^i to transform like a vector.

THEOREM 1.2. *The paths admit a continuous group with the maximum possible number $n(n+1)$ of essential parameters if and only if they are straight lines, the group then being necessarily the full linear group.*

Proof. The direct statement is obvious from $\ddot{x}^i = 0$ as the path equations. Combining the two portions of Theorem 1.1 gives the most general path-equations admitting the full linear group as $\ddot{x}^i+c\dot{x}^i = 0$ whose solutions are the straight lines $x^i = a^is+b^i$, $s = \exp(-ct)$, when $c \neq 0$. For the general converse, note that the maximum number of possible parameters in the solution of (3) is $n(n+1)$, corresponding to the number of initial conditions on u^i and $u^i{}_{,j}$. Each (independent) integrability condition reduces the number of these parameters by unity, so that the maximum number can be achieved if and only if these conditions are identically satisfied. From the lemma of the introductory section, we have the conditions as

$$L\epsilon^i = 0, \qquad LP^i_j = 0, \qquad L\alpha^i{}_{;j;k;l} = 0.$$

These must be satisfied identically in view of the properties of L. Therefore, $\alpha^i{}_{;j;k;l} = 0$, α^i being a polynomial of degree two in $\dot{x}$ (as in the Einstein–Mayer and similar relativity theories). Then

$$\alpha^i = \Gamma^i_{jk}\dot{x}^j\dot{x}^k + a^i_j\dot{x}^j + v^i,$$

whence $$L\Gamma^i_{jk} = 0, \qquad La^i_j = 0, \qquad Lv^i = 0.$$

Here (3) breaks up into the last three separate conditions, of which the last two are in fact included under $L\epsilon^i = 0$, and must be satisfied by the solutions of the first, which really form the group. Therefore v^i belongs to the centre of the group. There is no such possibility for a^i_j, which must be of form $a\delta^i_j$, while $P^i_j = b\delta^i_j$ by the same argument, a and b being at most invariants of the group, which would introduce at least one restriction unless both were constants. But the restriction upon P^i_j immediately gives $\alpha^i = q\dot{x}^i + ex^i + h^i$ where again a, e must be constants, and h^i a vector with constant components with $v^i = ex^i + h^i$. But this gives at once $u^i{}_{,j,k} = 0$, which makes the group a subgroup of the linear group. Inasmuch as all $n(n+1)$ parameters are essential by hypothesis, we can only have the full linear group, proving the theorem.

Because of the special interest of the Lorentz group in physics, we may consider the linear transformations, necessarily homogeneous, leaving a non-degenerate quadratic form invariant. This means the invariance of a symmetric tensor with constant coefficients a_{ij}, the group G_μ having then $\mu = \frac{1}{2}n(n-1)$ parameters. The invariants H and J of Theorem 1.1 must be built up of

$$X = a_{ij}x^ix^j, \qquad Y = a_{ij}\dot{x}^i\dot{x}^j, \qquad Z = a_{ij}\dot{x}^ix^j.$$

The rest of the technique is precisely that for the Lorentz group, as worked out elsewhere in full detail (**5**). It is seen at once that not all subgroups of the linear group are on the same footing after G_μ has been imposed. Calling similitudes S_1, we obtain

THEOREM 1.3. *The paths are straight lines if they admit $G_\mu + T_n$ and any other linear transformation; or if they admit $G_\mu + S_1$ and a translation. For $G_\mu + S_1$ and a homogeneous linear transformation, the path-equations reduce to the isotropic form $\ddot{x}^i + a\dot{x}^i + bx^i = 0$ (a, b constant).*

Proof. $G_\mu + T_n$ gives as paths $\ddot{x}^i + x^i f(Y) = 0$. The additional linear transformation immediately reduces $f(Y)$ to a constant. For $G_\mu + S_1$ we have the most general path-equations as

$$\ddot{x}^i + \dot{x}^i H(Y/X, Z/X) + x^i J(Y/X, Z/X) = 0.$$

Any translation admitted must make $J = 0$, $H = $ constant. For any other linear transformation, H, J may at most be constants.

This shows that, if a certain number of linear transformations be admitted, the whole of the homogeneous or full linear groups respectively will be admitted, the paths reducing to a very simple form. In the above theorem, for $G_\mu + T_n$, the additional continuous transformation group (one-parameter) may be taken as arbitrary, with the same final result. For the $G_\mu + S_1$ case, some generality is possible, as is seen from (**5**). However, if the space is endowed with a Riemann metric and isotropic in the sense of the theorem of F. Schur, it must then be flat, and the paths must be straight lines, admitting the whole linear group in a suitably chosen coordinate system. In this connexion, mention may be made of the recent work of G. Birkhoff (**6**), showing that isotropic Riemann spaces admitting the whole group for the entire manifold are restricted to the Euclidean and the classical non-Euclidean cases, the reason being that an isotropic Riemann space admits locally a group of $\frac{1}{2}n(n+1)$ parameters. Even in our own case, there would be considerable simplification and loss of generality if we demanded the absence of any singularities, for then terms like Y/X, Z/X could not be admitted in general without a singularity along the minimal lines and at the origin.

It follows from my work on the Lorentz group (**5** *b*) that there are Riemann spaces admitting a G_μ which are isotropic, and hence admit (locally) a group of dimension $\frac{1}{2}n(n+1)$, but are *not* flat.

2. The problem of classifying path-spaces according to the groups admitted cannot lead to very deep results in the general case, for it is already clear from the foregoing that any fairly complicated paths admit only the identity. If, to facilitate the consideration of the whole group, we restrict ourselves to formally expansible groups and α^i, then

$$\alpha^i = A^i + A^i_j \dot{x}^j + \Gamma^i_{jk}\dot{x}^j\dot{x}^k + A^i_{jkl}\dot{x}^j\dot{x}^k\dot{x}^l + \dots.$$

Here it will be clear, from the statement on differential invariants, at the end of the opening section, that only the quadratic terms behave like coefficients of connexion, the rest being symmetric tensors of the rank indicated. Inasmuch as each such tensor must vanish under the L operation, we have one additional restriction for each one not to be constructed from the others. The broadest groups must therefore be those that are restricted only by

$$L\Gamma^i_{jk} = 0, \qquad LR^i_{jkl} \equiv L\tfrac{1}{3}(P^i_{j;k} - P^i_{k;j})_l = 0,$$
$$A^i = 0, \qquad A^i_j = 0, \qquad A^i_{jkl} = 0, \dots. \tag{2.1}$$

The second sums up all the conditions of integrability. For the metric case, we have a very useful lemma in

THEOREM 2.1. *The paths $\ddot{x}^i+\Gamma^i_{jk}\dot{x}^i\dot{x}^j = 0$ are the geodesics of a metric of form $a_0+a_i\dot{x}^i+a_{ij}\dot{x}^i\dot{x}^j+a_{ijk}\dot{x}^i\dot{x}^j\dot{x}^k+...$ if and only if the first two terms form a perfect differential and the remaining tensor coefficients have a vanishing covariant derivative with respect to Γ^i_{jk}.*

The proof is by direct substitution of the series metric into the usual Euler equations, where the grouped terms of any given degree in $\dot{x}^i$ must vanish.

We have the corollary that for paths $\ddot{x}^i = 0$ the most general metric is one whose coefficients are constant tensors, being then of form $f(\dot{x})$. If we now impose upon the group preserving our path-space the further condition of leaving the metric invariant, we see that the quadratic term is preserved by at most $\frac{1}{2}n(n+1)$ transformations of the group, of which n are translations. To leave a general tensor of rank three invariant would be impossible for $n(n+1)(n+2)/6 > n(n+1)$, i.e. $n > 4$. Thus we see that *Riemannian geometry is the most general metric path-geometry possible in the sense of group theory*. Apart from cases of lower dimension, we should have invariance for the higher order terms only if they could be expressed by means of the second-order tensor and one or more vectors with vanishing covariant derivative, which again severely restricts the coefficients of connexion. For, if $T_{ijk..|m} = 0$, we have the curvature tensor restricted by

$$T_{rjk..}R^r_{ikm}+... = 0, \qquad \text{where} \quad R^i_{jkl} = \tfrac{1}{3}(P^i_{j;k}-P^i_{k;j})_{;l}.$$

Affinely connected spaces of two dimensions can be classified without much difficulty [Levine (**7**)]. For a single dimension we have a general

THEOREM 2.2. *An ordinary differential equation of the second order admits a one-parameter group if and only if it is of the form*

$$\ddot{x}+cu+c_2\dot{x}+\Gamma\dot{x}^2+c_3\dot{x}^3/u+c_4\dot{x}^4/u+... = 0;$$

it admits the maximal two-parameter group if and only if of the form $\ddot{x}+c\dot{x}+\Gamma\dot{x}^2 = 0$, where the c's are constants, $\Gamma = \Gamma(x)$, and the group is generated by one or two linear combinations respectively of

$$\exp\Big(-\int\Gamma\Big) \quad \textit{and} \quad \exp\Big(-\int\Gamma\Big)\int\exp\Big(\int\Gamma\Big).$$

The theorem is to be proved by direct substitution. The difference from the general case is due really to the fact that, for $n = 1$, P^i_j is always of form $P\delta^i_j$. A canonical form for this case may be obtained by taking the first transformation as a translation. Then the equation

becomes $\ddot{x}+f(\dot{x}) = 0$. If there is another non-zero solution $u(x)$ of the equations of variation $u''\dot{x}^2-u'f(\dot{x})+u'\dot{x}f'(\dot{x}) = 0$, clearly f must be of the form $a\dot{x}^2+b\dot{x}$. The second group generator is then $\exp(-ax)$, unless $a = 0$, in which case it is x; b remains an arbitrary constant. Taking $\bar{x} = \exp(ax)$ makes the 2-parameter group the full linear group.

3. In the general case, let us concentrate only upon the symmetric affine connexions, as in (2.1) because they admit groups of maximal orders. For any purely covariant 3-index tensor, we have the following well-known (**8**) irreducible decomposition

$$T_{ijk} = T_{(ijk)}+T_{[ijk]}+\tfrac{2}{3}\{T_{[ij]k}-T_{[kj]i}\}+\tfrac{2}{3}\{T_{(ij)k}-T_{k(ji)}\}, \tag{3.1}$$
$$n^3 = \tfrac{1}{6}n(n+1)(n+2)+\tfrac{1}{6}n(n-1)(n-2)+\tfrac{1}{3}n(n^2-1)+\tfrac{1}{3}n(n^2-1).$$

Here and hereafter, the round and square brackets in subscript denote the completely symmetric and completely antisymmetric portions of the enclosed tensor-indices. The second line of the formula gives the actual reduced numbers in each component. For the curvature tensor R^i_{jkl} we have the identities

$$R^i_{(jk)l} = 0, \qquad R^i_{jkl}+R^i_{klj}+R^i_{ljk} = 0. \tag{3.2}$$

Therefore, as regards the covariant indices, the curvature tensor is *irreducible*, the first two components in (3.1) vanishing identically while the last two are equivalent but for change of indices. Thus, the number of determining elements of the tensor is $\frac{1}{3}n^2(n^2-1)$, which is therefore the general number of compatibility conditions for $L\Gamma^i_{jk} = 0$. Therefore, *the most general path-spaces with symmetric affine connexion admit only the identity for $n > 2$.* Noting that the reducibility properties of $LT^{\cdot\cdot}_{\cdot\cdot}$ are the same as those of $T^{\cdot\cdot}_{\cdot\cdot}$ itself, we have to take the upper index into account also. Here, we can form a separate component with δ^i_j and any contraction of R^i_{jkl}. Because of (3.2), we may take $R_{ij} = R^r_{rij}$ as the contraction in terms of which the other two possible contractions can be expressed. Preserving the given symmetry properties of the tensor, we get the final reduction as

$$R^i_{jkl} = W^i_{jkl}+\delta^i_j\left\{\frac{1}{n-1}R_{(kl)}+\frac{1}{n+1}R_{[kl]}\right\}-\frac{2}{n+1}\delta^i_l R_{[jk]}-$$
$$-\delta^i_k\left\{\frac{1}{n-1}R_{(jl)}+\frac{1}{n+1}R_{[jl]}\right\}: \tag{3.3}$$

this may be used as a definition of W^i_{jkl}, which, known as *the Weyl tensor, is clearly irreducible.* Its vanishing means that the space is projectively flat; and it is unchanged when the affine connexion is replaced by
$$\bar{\Gamma}^i_{jk} = \Gamma^i_{jk}+\delta^i_j\psi_k+\delta^i_k\psi_j.$$

If we ask that the Lie group represent such 'projective' collineations, the compatibility conditions obtained by eliminating the arbitrary ψ_i from

$$L\Gamma^i_{jk} = \delta^i_j \psi_k + \delta^i_k \psi_j$$

become merely $LW^i_{jkl} = 0$, plus further equations obtained by covariant differentiation of W.

We restrict ourselves to the projectively flat spaces $W^i_{jkl} = 0$, seeing that groups with maximum number of parameters are under investigation. We may then use the abbreviation

$$R^i_{jkl} = \delta^i_j B_{kl} - \delta^i_k B_{jl} - 2\delta^i_l B_{[jk]} \tag{3.4}$$

with

$$B_{ij} = \frac{1}{n+1} R_{[ij]} + \frac{1}{n-1} R_{(ij)}.$$

For all curvature tensors, we have the identity

$$R^i_{jkl|m} + R^i_{kml|j} + R^i_{mjl|k} = 0. \tag{3.5}$$

These two, with suitable contractions of the upper with a lower index, give

$$B_{ij|k} - B_{kj|i} = 0. \tag{3.6}$$

Cyclic rotation of the indices ($n > 2$) gives at once

$$B_{[ij]|k} + B_{[jk]|i} + B_{[ki]|j} = 0.$$

This well-known result says that the exterior derivative of $B_{[ij]}$ (hence of $R_{[ij]}$) must vanish, the antisymmetric portion of the contracted curvature tensor being a curl.

The number of compatibility conditions is still too great unless the covariant derivatives vanish identically, $R_{ij|k} = 0$, which we now assume. It would seem at first sight that there are then three cases, according as (i) $R_{ij} = 0$, which gives the flat spaces of § 1 with straight lines for paths; (ii) $R_{(ij)} = 0$, which would give groups of dimension $\frac{1}{2}n(n+3)$; and (iii) $R_{[ij]} = 0$, which would preserve the symmetric tensor giving groups of dimension $\frac{1}{2}n(n+1)$. The middle, purely antisymmetric case cannot arise. The vanishing of $R_{ij|k}$ gives again its own compatibility conditions by alternating the second covariant derivatives. These reduce to

$$B_{lj} B_{[ik]} + B_{kj} B_{[li]} - 2B_{ij} B_{[kl]} = 0. \tag{3.7}$$

Interchange of i, k and addition gives a further reduction

$$B_{kj} B_{[li]} = B_{ij} B_{[kl]} = B_{lj} B_{[ik]}. \tag{3.8}$$

This shows that *in general*

$$B_{ij} B_{[kl]} = 0, \tag{3.9}$$

i.e. the vanishing of the covariant derivative entails the symmetry of the contracted curvature tensor for general projectively flat spaces.

THEOREM 3.1. *A projectively flat space* ($n > 3$) *with vanishing covariant derivative for the curvature tensor is either flat and admits the full linear group, or, if* $|R_{ij}| \neq 0$, *admits a group of* $\frac{1}{2}n(n+1)$ *parameters, being Riemannian in the latter case.*

If a space with symmetric affine connexion admit T_n, then clearly the Γ^i_{jk} are constants for the given coordinate system. In this case, we have

$$R^i_{jkl} = \Gamma^i_{rk}\Gamma^r_{jl} - \Gamma^i_{rj}\Gamma^r_{kl}, \qquad R_{ij} = \Gamma^k_{ri}\Gamma^r_{kj} - \Gamma^r_{rk}\Gamma^k_{ij},$$

so that R_{ij} *is symmetric regardless of the vanishing of the covariant derivative and the dimension if the path-space admit* T_n.

For a single one-parameter group we may, as stated, choose new coordinates making the group into a translation, say along the x^1 direction. If there exists another generating vector $u^i(x)$ which permutes with the first $Lu^i = 0$, then x^1 cannot enter into u^i. Therefore another transformation of coordinates preserving x^1 can always be found (locally at least), making the group of $u^i(x)$ a translation along x^2, choosing the x^2 coordinates along the trajectories defined by

$$dx^i/ds = u^i(x).$$

The process may be continued to a group of not more than n dimensions provided that all the operators commute, the Lie constants of composition being zero. Therefore our theorems remain valid when T_n is replaced by an n-parameter Abelian group, provided, of course, that corresponding changes are made in statements involving other special groups.

Inasmuch as the maximum possible number of parameters for groups preserving the paths is just twice that for groups preserving a Riemann metric, it follows that there may exist transformations preserving the former but not the latter. The general theorem (**1**) is that a metric is transformed into another by any collineation. For affinely connected spaces, this takes on the special form: *The covariant derivative of a tensor vanishes after a transformation of a continuous group if and only if it vanishes originally.*

The proof follows directly from our lemma, for $L(T^{\cdot\cdot}_{\cdot\cdot|i}) = (LT^{\cdot\cdot}_{\cdot\cdot})_{|i}$; whence both vanish together whether or not $LT^{\cdot\cdot}_{\cdot\cdot} = 0$. For a Riemann metric and an analytic group, $|g_{ij}|$ will not vanish for some range of values of the parameter about the identity, for the transform of this determinant may be directly expanded in series, using the definition of L for relative tensor-invariants. Thus the metric tensors associated

with a given set of paths provide a method for the study of the collineation group and its subgroups. The non-removable singularities of the space furnish another approach, for the group must either leave the singular points invariant, or transform equivalent singularities into each other; and this applies to the whole group, which may thus contain a non-expansible (finite) sub-group when the singularities are discrete.

4.† The complete classification of path-spaces with respect to their collineation groups can be greatly simplified by the following considerations.

THEOREM 4.1. *The most general path-spaces admitting a given (maximal) Lie group are found by taking any space of symmetric affine connexion* ($\alpha^i = \Gamma^i_{jk}\dot{x}^j\dot{x}^k$) *admitting that group and adding to that* α^i *the most general contravariant vector* $\lambda^i(x, \dot{x})$ *preserved by that group.*

This extends Theorem 1.1.

Proof. For given n, the difference of any two α^i is a contravariant vector under transformation of point coordinates, as is seen directly from the transformation law for α^i. Thus to any α^i preserved by a group, any arbitrary contravariant vector may be added which is invariant under that group. If the α^i be formally expansible in power series in $\dot{x}$, we have seen that the second degree terms give a symmetric affine connexion, all the rest having tensor coefficients, hence amounting to an additive vector. Now $\epsilon^i = \alpha^i - \frac{1}{2}\alpha^i_{;r}\dot{x}^r$ is a differential invariant of the space wherein that portion of α^i homogeneous of degree two in $\dot{x}$ does not appear. Inasmuch as the group preserves all differential invariants, it suffices to treat this as a contribution of the additive vector, and to deal only with the second degree homogeneous portion. Upon point transformation, there will be added thereto a quadratic form in $\dot{x}$; hence we may separate this homogeneous term into two additive components, a quadratic form and a vector. But this set of quadratic forms (which may be zero) brings us to the affine connexions, proving the theorem. It should be noted that the group need not be maximal for the affine connexion.

THEOREM 4.2. *For* $n = 2$, *the collineation groups of the path-space must necessarily be sub-groups of the projective group in two variables (for suitable choice of variables).*

Proof. From the preceding, it suffices to prove the theorem for 2-dimensional affine connexions; but here it really follows from a result

† § 4 is an addition by the author which was received on 26 July 1951.

of Lie–Engel [*Theorie d. Transformationsgruppen*, iii, 76]. Levine (7) seems to have ignored this, and gives two groups for affine connexions that are not immediately seen to be sub-groups of the projective group. In the standard notation $p = \partial/\partial x$, $q = \partial/\partial y$, these are generated by the operators: p, q, yq, $e^x q$; and p, yq, $(e^{ax}\cos x)q$, $(e^{ax}\sin x)q$ respectively. The first transforms to xp, q, yq, xq by taking $x = \log \bar{x}$; the second to $(1+x^2)p+xyq$, yq, q, xq by taking $\bar{x} = \tan x$, $\bar{y} = ye^{-ax}\sec x$. This gives a direct proof and classification of the case $n = 2$ with minimum labour.

This property of two-dimensional path-spaces extends in general to the projectively flat path-spaces, but this will be considered elsewhere because new definitions for 'totally geodesic' subspaces of a path-space are needed which would take us away from the immediate result indicated.

REFERENCES

1. D. D. Kosambi, 'Lie rings in path-space', *Proc. National Acad. Sci.* (U.S.A.) 35 (1949), 389–94.
2. —— 'Systems of differential equations of higher order', *Quart. J. of Math.* (Oxford) 6 (1935), 1–12.
3. —— 'Series expansions of continuous groups', ibid. (2) 2 (1951), 244–57.
4. K. Yano, *Groups of transformations in generalized spaces* (Tokyo, 1949); 'Sur la théorie des déformations infinitésimales', *J. Fac. Sci. Univ. Tokyo*, 6 (1949), 1–76.
5. D. D. Kosambi, 'Path-equations admitting the Lorentz group,' (*a*), *J. of London Math. Soc.* 15 (1940), 86–91; (*b*), (II), *J. Indian Math. Soc.* 5 (1941), 62–72.
6. Garrett Birkhoff, 'Extensions of Lie groups', *Math. Zeitschrift*, 53 (1950), 225–35.
7. J. Levine, 'Classification of collineations in projectively and affine connected spaces of two dimensions', *Ann. Math.* 52 (1950), 465–77.
8. J. A. Schouten, *Der Ricci-Kalkül* (Berlin, 1924).

A RESULT IN HILBERT SPACE

By N. A. ROUTLEDGE (*Cambridge*)

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1. THIS paper is concerned with the problem of finding the smallest sphere which will contain a given set of points. The problem has been solved in the case of n-dimensional Euclidean space [see T. Bonnesen and W. Fenchel, 'Theorie der konvexen Körper', *Ergebnisse der Math.* (Berlin, 1934), 78, and also the references given there]. Here I extend the result to both real and complex Hilbert space, showing that for any set of diameter δ there is a sphere of radius $\delta/\sqrt{2}$ containing it.

2. Complex Hilbert space consists of all infinite sequences of complex numbers $\{x_n\}$, which I shall also denote by $\mathbf{x}$, such that

$$\sum_{n=1}^{\infty} |x_n|^2 < \infty.$$

The distance between two points $\mathbf{x}$ and $\mathbf{y}$ is

$$\|\mathbf{x}-\mathbf{y}\| = \sqrt{\sum_{n=1}^{\infty} |x_n-y_n|^2},$$

which is easily proved to be finite. When we have several points we may denote them by ${}_1\mathbf{a}$, ${}_2\mathbf{a}$, ${}_3\mathbf{a}$..., these consisting of the sequences

$$\{{}_1a_n\},\ \{{}_2a_n\},\ldots.$$

The real Hilbert space is obtained by restricting the sequences to be of *real* numbers.

A sphere, of course, is given by specifying its centre $\mathbf{a}$ and its radius r, and consists simply of those points $\mathbf{x}$ such that

$$\|\mathbf{x}-\mathbf{a}\| \leqslant r;$$

and the diameter of a set is ubd $\|\mathbf{x}-\mathbf{y}\|$ for all points $\mathbf{x}$, $\mathbf{y}$ of the set.

3. THEOREM 1. *Given, in n-dimensional Euclidean space, a non-empty set of finite diameter δ, then there is a unique sphere of minimum radius r (say) containing it, where*

$$r \leqslant \delta\sqrt{\frac{n}{2(n+1)}}.$$

Further, this is the best possible result.

THEOREM 2. *If ${}_1\mathbf{a}$, ${}_2\mathbf{a}$,..., ${}_m\mathbf{a}$ are points in real (or complex) Hilbert*

space, then, if $\mathbf{b}$ *is the point* $\mathbf{b} = ({}_1\mathbf{a}+{}_2\mathbf{a}+...+{}_m\mathbf{a})/m$ (*the* CENTRE OF GRAVITY) *and* $\mathbf{x}$ *is any point,*

$$\sum_{r=1}^{m} \|\mathbf{x}-{}_r\mathbf{a}\|^2 = m\,\|\mathbf{x}-\mathbf{b}\|^2+\sum_{r=1}^{m} \|\mathbf{b}-{}_r\mathbf{a}\|^2.$$

Theorem 1 is the result already known and quoted above, and Theorem 2, when the definition of $\|\mathbf{x}-\mathbf{y}\|$ is written in it, is a simple identity. I shall refer to this as *the theorem of Apollonius*, since it is an extension of that theorem in elementary geometry.

It is Theorem 1 that I wish to extend to Hilbert space.

4. I shall now, until further notice, restrict myself to the *real* space.

Denote by $\mathfrak{M}_m$ the subspace of all points $\mathbf{x}$ such that $x_n = 0$ when $n > m$. Then the *projection* of any point $\mathbf{x}$ onto $\mathfrak{M}_m$ is defined to be the point $\mathbf{u}$ where

$$u_n = \begin{cases} x_n & (n \leqslant m), \\ 0 & (n > m). \end{cases} \tag{1}$$

LEMMA 3. *When we keep* $\|\mathbf{x}-\mathbf{y}\|$ *as the distance,* $\mathfrak{M}_m$ *is an m-dimensional Euclidean space.*

This is readily seen.

LEMMA 4. *If* $\mathbf{u}$ *and* $\mathbf{v}$ *are the projections of* $\mathbf{x}$ *and* $\mathbf{y}$ *onto* $\mathfrak{M}_m$, *then*

$$\|\mathbf{u}-\mathbf{v}\| \leqslant \|\mathbf{x}-\mathbf{y}\|.$$

For

$$\begin{aligned} \|\mathbf{u}-\mathbf{v}\|^2 &= \sum_{n=1}^{\infty} |u_n-v_n|^2 \\ &= \sum_{n=1}^{m} |x_n-y_n|^2, \text{ by the definition of projection,} \\ &\leqslant \sum_{n=1}^{\infty} |x_n-y_n|^2 \\ &= \|\mathbf{x}-\mathbf{y}\|^2. \end{aligned}$$

LEMMA 5. *If* $m_1 \geqslant m_2$, *the projection on* $\mathfrak{M}_{m_2}$ *of the projection of a point onto* $\mathfrak{M}_{m_1}$ *is the projection of that point on* $\mathfrak{M}_{m_2}$.

This should be obvious.

Consider now a non-empty set, of finite diameter δ. Let $\mathfrak{A}_m$ be its projection onto $\mathfrak{M}_m$ (i.e. the set of points which are projections of points of $\mathfrak{A}$). Then, by Lemma 4, if the diameter of $\mathfrak{A}_n$ is δ_n,

$$\delta_n \leqslant \delta. \tag{2}$$

Thus δ_n is finite, and so, by Lemma 3, Theorem 1, and (2), we have

LEMMA 6. *In $\mathfrak{M}_m$ there is a unique sphere of minimum radius containing $\mathfrak{A}_m$, and, if this sphere has centre ${}_m\mathbf{a}$ and radius r_m,*

$$r_m \leqslant \delta_m \sqrt{\frac{m}{2(m+1)}} \leqslant \delta/\sqrt{2}.$$

LEMMA 7. *The sequence r_m is an increasing sequence.*

For, if $m_1 \geqslant m_2$, let $\mathbf{d}$ be the projection of ${}_{m_1}\mathbf{a}$ onto $\mathfrak{M}_{m_2}$ and let $\mathbf{x}$ be the general point of $\mathfrak{A}_{m_1}$ and let $\mathbf{u}$ be the projection of $\mathbf{x}$ on $\mathfrak{M}_{m_2}$. Then, by Lemma 5, $\mathbf{u}$ is the general point of $\mathfrak{A}_{m_2}$. Now

$$\begin{aligned} \|\mathbf{d}-\mathbf{u}\| &\leqslant \|{}_{m_1}\mathbf{a}-\mathbf{x}\|, \text{ by Lemma 4,} \\ &\leqslant r_{m_1} \end{aligned}$$

since, by Lemma 6, the sphere with centre ${}_{m_1}\mathbf{a}$ and radius r_{m_1} contains $\mathfrak{A}_{m_1}$. Hence the sphere with centre $\mathbf{d}$ and radius r_{m_1} contains $\mathfrak{A}_{m_2}$. Hence r_{m_2}, being the minimum radius of spheres in $\mathfrak{M}_{m_2}$ containing $\mathfrak{A}_{m_2}$, has $r_{m_2} \leqslant r_{m_1}$. This is Lemma 7.

Thus, by Lemmas 6 and 7, $\{r_m\}$ is an increasing bounded sequence. Hence it tends to a limit as $m \to \infty$.

Let $$\lim_{m\to\infty} r_m = r \leqslant \delta/\sqrt{2}, \text{ by Lemma 6.} \tag{3}$$

LEMMA 8. $\lim_{m\to\infty} {}_m a_n$ *exists for each n, and, if*

$$a_n = \lim_{m\to\infty} {}_m a_n,$$

then

$$\sum_{n=1}^{\infty} a_n^2 < \infty,$$

i.e. $\{a_n\} = \mathbf{a}$ is a point of the space.

Given $\epsilon > 0$, by (3) find M so that

$$r^2-\epsilon \leqslant r_m^2 \leqslant r^2 \quad \text{for all } m \geqslant M. \tag{4}$$

If $m_1 \geqslant m_2$, taking the same nomenclature as in Lemma 7, we have, by the Theorem of Apollonius,

$$\begin{aligned} &\|\mathbf{u}-\mathbf{d}\|^2+\|\mathbf{u}-{}_{m_2}\mathbf{a}\|^2 \\ &\quad = 2\|\mathbf{u}-\tfrac{1}{2}(\mathbf{d}+{}_{m_2}\mathbf{a})\|^2+\|\tfrac{1}{2}(\mathbf{d}+{}_{m_2}\mathbf{a})-\mathbf{d}\|^2+\|\tfrac{1}{2}(\mathbf{d}+{}_{m_2}\mathbf{a})-{}_{m_2}\mathbf{a}\|^2 \\ &\quad = 2\|\mathbf{u}-\tfrac{1}{2}(\mathbf{d}+{}_{m_2}\mathbf{a})\|^2+\tfrac{1}{2}\|\mathbf{d}-{}_{m_2}\mathbf{a}\|^2. \end{aligned}$$

So

$$\begin{aligned} \|\mathbf{u}-\tfrac{1}{2}(\mathbf{d}+{}_{m_2}\mathbf{a})\|^2 &= \tfrac{1}{2}\|\mathbf{u}-\mathbf{d}\|^2+\tfrac{1}{2}\|\mathbf{u}-{}_{m_2}\mathbf{a}\|^2-\tfrac{1}{4}\|\mathbf{d}-{}_{m_2}\mathbf{a}\|^2 \\ &\leqslant \tfrac{1}{2}\|\mathbf{x}-{}_{m_1}\mathbf{a}\|^2+\tfrac{1}{2}\|\mathbf{u}-{}_{m_2}\mathbf{a}\|^2-\tfrac{1}{4}\|\mathbf{d}-{}_{m_2}\mathbf{a}\|^2, \text{ by Lemma 4,} \\ &\leqslant \tfrac{1}{2}r_{m_1}^2+\tfrac{1}{2}r_{m_2}^2-\tfrac{1}{4}\|\mathbf{d}-{}_{m_2}\mathbf{a}\|^2, \text{ by Lemma 6,} \\ &\leqslant r^2-\tfrac{1}{4}\|\mathbf{d}-{}_{m_2}\mathbf{a}\|^2, \text{ by (3) and Lemma 7.} \end{aligned}$$

Remarking that $\mathbf{u}$ is the general point of $\mathfrak{A}_{m_2}$ and that $\tfrac{1}{2}(\mathbf{d}+{}_{m_2}\mathbf{a})$ is in

$\mathfrak{M}_{m_2}$, this says that $\mathfrak{A}_{m_2}$ can be contained, in $\mathfrak{M}_{m_2}$, in a sphere of a certain radius. So, by definition of r_{m_2},

$$r_{m_2}^2 \leqslant r^2 - \tfrac{1}{4}\|\mathbf{d} - {}_{m_2}\mathbf{a}\|^2.$$

Consequently, by (4), $\qquad \|\mathbf{d} - {}_{m_2}\mathbf{a}\|^2 \leqslant 4\epsilon.$

Therefore, recalling the definition of $\mathbf{d}$ in Lemma 7, and also that $\mathbf{d}$ and ${}_{m_2}\mathbf{a}$ are in $\mathfrak{M}_{m_2}$, so that we have $d_n = 0$, ${}_{m_2}a_n = 0$ $(n > m_2)$, we see that

$$\sum_{n=1}^{m_2} |d_n - {}_{m_2}a_n|^2 \leqslant 4\epsilon,$$

i.e.

$$\sum_{n=1}^{m_2} |{}_{m_1}a_n - {}_{m_2}a_n|^2 \leqslant 4\epsilon.$$

Thus

$$|{}_{m_1}a_n - {}_{m_2}a_n| \leqslant 2\surd\epsilon \quad (n \leqslant m_2).$$

The only assumption was that $m_2 \geqslant m_1 \geqslant M$. Hence, for fixed n,

$$\lim_{m\to\infty} {}_m a_n \text{ exists } (= a_n \text{ say}). \tag{5}$$

We have now to prove that

$$\sum_{n=1}^{\infty} a_n^2 < \infty.$$

Now $\mathfrak{A}$ is not empty. Let $\mathbf{y}$ be one of its points. Then, from Lemma 6, if $\mathbf{z}$ is the projection of $\mathbf{y}$ on $\mathfrak{M}_m$,

$$\|\mathbf{z} - {}_m\mathbf{a}\|^2 \leqslant r_m^2 \leqslant \tfrac{1}{2}\delta^2,$$

i.e.

$$\sum_{r=1}^{m} |y_r - {}_m a_r|^2 \leqslant \tfrac{1}{2}\delta^2.$$

But $\displaystyle \sum_{r=1}^{m} |{}_m a_r|^2 = \sum_{r=1}^{m} |y_r - {}_m a_r - y_r|^2 \leqslant 4\sum_{r=1}^{m} |y_r - {}_m a_r|^2 + 4\sum_{r=1}^{m} y_r^2.$

(This follows from $(a-b)^2 \leqslant 4a^2 + 4b^2$, which is easily proved.)

Accordingly

$$\sum_{r=1}^{m} {}_m a_r^2 \leqslant 4.\tfrac{1}{2}\delta^2 + 4\sum_{r=1}^{m} y_r^2$$

$$\leqslant 2\delta^2 + 4\sum_{r=1}^{\infty} y_r^2.$$

Hence

$$\sum_{r=1}^{m_1} {}_{m_2} a_r^2 \leqslant 2\delta^2 + 4\sum_{r=1}^{\infty} y_r^2 \quad \text{if } m_1 \leqslant m_2.$$

Let $m_2 \to \infty$. It follows from (5) that

$$\sum_{r=1}^{m_1} a_r^2 \leqslant 2\delta^2 + 4\sum_{r=1}^{\infty} y_r^2,$$

and so

$$\sum_{r=1}^{\infty} a_r^2 \leqslant 2\delta^2 + 4\sum_{r=1}^{\infty} y_r^2 < \infty,$$

since $\mathbf{y}$ is in the Hilbert space, and by § 2 the condition for this was

$\sum y_r^2 < \infty$. This completes the proof of the lemma. We can now establish the main result of the paper for the real space.

THEOREM 9. *If $\mathfrak{A}$ is a non-empty set, of finite diameter δ, in real Hilbert space, then there is a unique sphere of minimum radius containing $\mathfrak{A}$, and this radius does not exceed $\delta/\sqrt{2}$.*

Further, this constant is the best possible.

Indeed the required sphere is the one of centre $\mathbf{a}$ and radius r, where $\mathbf{a}$ and r are as in Lemma 8 and (3). I show this by a series of lemmas, recalling from (3) that $r \leqslant \delta/\sqrt{2}$.

LEMMA 10. *The sphere with centre* $\mathbf{a}$ *and radius r contains* $\mathfrak{A}$.

Let $\mathbf{x}$ be any point of $\mathfrak{A}$. Let $\mathbf{z}$ be its projection on $\mathfrak{M}_m$. Then, by Lemma 6

$$\|\mathbf{z}-{}_m\mathbf{a}\|^2 \leqslant r_m^2$$
$$\leqslant r^2, \text{ by (3) and Lemma 7,}$$

i.e.
$$\sum_{n=1}^{m} |x_n - {}_m a_n|^2 \leqslant r^2.$$

Thus
$$\sum_{n=1}^{m_1} |x_n - {}_{m_2} a_n|^2 \leqslant r^2 \quad (m_1 \leqslant m_2).$$

Let $m_2 \to \infty$. Then

$$\sum_{n=1}^{m_1} |x_n - a_n|^2 \leqslant r^2, \text{ by Lemma 8,}$$

and so
$$\sum_{n=1}^{\infty} |x_n - a_n|^2 \leqslant r^2,$$

i.e.
$$\|\mathbf{x}-\mathbf{a}\|^2 \leqslant r^2.$$

This proves the lemma.

LEMMA 11. *No sphere of radius less than r can contain* $\mathfrak{A}$.

For, if the sphere with centre $\mathbf{b}$ and radius $s < r$ contains $\mathfrak{A}$, then, since $\lim_{m\to\infty} r_m = r$, we can find m_0 so that

$$r_{m_0} > s. \tag{6}$$

Let $\mathbf{c}$ be the projection of $\mathbf{b}$ on $\mathfrak{M}_{m_0}$. Let $\mathbf{x}$ be the general point of $\mathfrak{A}$, and $\mathbf{u}$ the projection of $\mathbf{x}$.

Then, by Lemma 4,

$$\|\mathbf{u}-\mathbf{c}\| \leqslant \|\mathbf{x}-\mathbf{b}\| \leqslant s, \text{ by hypothesis.}$$

Thus $\mathfrak{A}_{m_0}$ can be contained in a sphere with centre $\mathbf{c}$ and radius s. Hence, by definition of r_{m_0}, $r_{m_0} \leqslant s$, which contradicts (6).

LEMMA 12. *No other sphere of radius r can contain* $\mathfrak{A}$.

For let the sphere with centre $\mathbf{b}$ and radius r contain $\mathfrak{A}$. I shall show that $\mathbf{b} = \mathbf{d}$. For, if $\mathbf{x}$ is any point of $\mathfrak{A}$,

$$\|\mathbf{x}-\mathbf{a}\|^2+\|\mathbf{x}-\mathbf{b}\|^2 = 2\|\mathbf{x}-\tfrac{1}{2}(\mathbf{a}+\mathbf{b})\|^2+\|\tfrac{1}{2}(\mathbf{a}+\mathbf{b})-\mathbf{a}\|^2+\|\tfrac{1}{2}(\mathbf{a}+\mathbf{b})-\mathbf{b}\|^2,$$

by the theorem of Apollonius. So

$$\begin{aligned}\|\mathbf{x}-\tfrac{1}{2}(\mathbf{a}+\mathbf{b})\|^2 &= \tfrac{1}{2}\|\mathbf{x}-\mathbf{a}\|^2+\tfrac{1}{2}\|\mathbf{x}-\mathbf{b}\|^2-\tfrac{1}{4}\|\mathbf{a}-\mathbf{b}\|^2\\ &\leqslant \tfrac{1}{2}r^2+\tfrac{1}{2}r^2-\tfrac{1}{4}\|\mathbf{a}-\mathbf{b}\|^2,\end{aligned}$$

by Lemma 10 and by hypothesis. Therefore $\mathfrak{A}$ can be contained in a sphere of radius

$$\sqrt{(r^2-\tfrac{1}{4}\|\mathbf{a}-\mathbf{b}\|^2)}.$$

Accordingly, by Lemma 11,

$$r^2 \leqslant r^2-\tfrac{1}{4}\|\mathbf{a}-\mathbf{b}\|^2.$$

Then

$$\|\mathbf{a}-\mathbf{b}\|^2 \leqslant 0,$$

i.e.

$$\sum_{n=1}^{\infty}|a_n-b_n|^2 \leqslant 0,$$

and so $a_n = b_n$ for all n; that is $\mathbf{a} = \mathbf{b}$.

Theorem 9 will thus be established if we can show that there is a set $\mathfrak{A}$ such that

$$r = \delta/\sqrt{2}.$$

This is done below in § 5.

5. Consider the set of points ${}_1\mathbf{u}$, ${}_2\mathbf{u}$, ${}_3\mathbf{u}$,..., where ${}_m\mathbf{u}$ is the series $\{{}_mu_n\}$, such that

$${}_mu_n = \begin{cases} 0 & (n \neq m),\\ \delta/\sqrt{2} & (n = m).\end{cases}$$

What is the diameter of this set?

Now

$$\|{}_{m_1}\mathbf{u}-{}_{m_2}\mathbf{u}\|^2 = \begin{cases} 0 & (m_1 = m_2),\\ \delta^2 & (m_1 \neq m_2),\end{cases}$$

by an easy evaluation. Hence the diameter is δ. Now, by the theorem of Apollonius,

$$\sum_{r=1}^{m}\|\mathbf{a}-{}_r\mathbf{u}\|^2 = m\|\mathbf{a}-\mathbf{b}\|^2 + \sum_{r=1}^{m}\|\mathbf{b}-{}_r\mathbf{u}\|^2,$$

where $\mathbf{b} = ({}_1\mathbf{u}+{}_2\mathbf{u}+...+{}_m\mathbf{u})/m$ and $\mathbf{a}$ is defined as before. Thus

$$\sum_{r=1}^{m}\|\mathbf{a}-{}_r\mathbf{u}\|^2 = m\|\mathbf{a}-\mathbf{b}\|^2+\sum_{r=1}^{m}\left[(m-1)\left(\frac{1}{m}\right)^2+\left(1-\frac{1}{m}\right)^2\right]\frac{\delta^2}{2},$$

on evaluation,

$$= m\|\mathbf{a}-\mathbf{b}\|^2+\tfrac{1}{2}(m-1)\delta^2 \geqslant \tfrac{1}{2}(m-1)\delta^2.$$

But, by Lemma 10,

$$\|\mathbf{a}-{}_r\mathbf{u}\|^2 \leqslant r^2.$$

Thus

$$r^2 \geqslant \tfrac{1}{2}(1-1/m)\,\delta^2.$$

This is true for all m. Hence $r^2 \geqslant \frac{1}{2}\delta^2$. But, by (3), $r^2 \leqslant \frac{1}{2}\delta^2$. We therefore have $r = \delta/\sqrt{2}$.

This completes the proof of Theorem 9.

6. This result extends immediately to the complex space by the use of a certain (1, 1) correspondence.

A point $\mathbf{x}$ of real Hilbert space, $\mathfrak{H}_1$ say, corresponds to a point $\mathbf{x}^*$ of complex Hilbert space, $\mathfrak{H}_2$ say, if

$$x_{2n} = y_n, \qquad x_{2n-1} = z_n \quad (n = 1, 2,...), \tag{7}$$

where $x_n^* = y_n + iz_n$ (real and imaginary parts). This is clearly a (1, 1) correspondence. Further we have

LEMMA 13. *This correspondence preserves distance.*

For
$$\begin{aligned} \|{}_1\mathbf{x}^* - {}_2\mathbf{x}^*\|^2 &= \sum_{n=1}^{\infty} |{}_1x_n^* - {}_2x_n^*|^2 \\ &= \sum_{n=1}^{\infty} |({}_1y_n + i\,{}_1z_n) - ({}_2y_n + i\,{}_2z_n)|^2 \\ &= \sum_{n=1}^{\infty} ({}_1y_n - {}_2y_n)^2 + ({}_1z_n - {}_2z_n)^2 \\ &= \sum_{n=1}^{\infty} ({}_1x_n - {}_2x_n)^2, \text{ by (7)}, \\ &= \|{}_1\mathbf{x} - {}_2\mathbf{x}\|^2. \end{aligned}$$

Finally we have

THEOREM 14. *Theorem* 9 *also holds for complex Hilbert space.*

For, given a set $\mathfrak{A}^*$ in $\mathfrak{H}_2$ of diameter δ, then, if $\mathfrak{A}$ is the corresponding set in $\mathfrak{H}_1$ we see from Lemma 13 that $\mathfrak{A}$ is of diameter δ. Hence by Theorem 9, there is a unique sphere of minimum radius containing it, of centre $\mathbf{a}$ and radius r say, such that $r \leqslant \delta/\sqrt{2}$. Hence, by Lemma 13, the sphere in $\mathfrak{H}_2$ of centre $\mathbf{a}^*$ and radius r contains $\mathfrak{A}^*$. Further, no smaller sphere, or sphere of radius r and centre other than $\mathbf{a}^*$ can contain $\mathfrak{A}^*$; for then the corresponding sphere in $\mathfrak{H}_1$ would contain $\mathfrak{A}$, which, by Theorem 9, is not the case. Hence the sphere of centre $\mathbf{a}^*$ and radius r is the unique sphere in $\mathfrak{H}_2$ of minimum radius containing $\mathfrak{A}^*$.

Moreover, the set in $\mathfrak{H}_2$ which corresponds to the set constructed in § 5 has, by Lemma 13, the radius $r = \delta/\sqrt{2}$.

This shows the constant to be best possible and completes the proof of Theorem 14, and also this paper.

Note. Theorem 9 can, of course, be proved more simply using the Bolzano–Weierstrass Theorem, and the 'axiom of choice'. This paper should perhaps be read as a pendant of 'A proof... in Hilbert space, without use of the axiom of choice' by I. Barsotti (*Bulletin of the American Math. Soc.* 53 (1947), 943–9), and furnishes a sharpening of his Lemma 1.

RIEMANN EXTENSIONS

By E. M. PATTERSON *and* A. G. WALKER (*Sheffield*)

[Received 21 December 1950]

1. Introduction

THE present paper continues the study, made in a recent series of papers,† of Riemannian spaces which admit parallel fields of partially null planes. A case of particular interest is that of a $2n$-space R^{2n} which admits a parallel field of null n-planes generated by a symmetric second-order recurrent tensor. In this case we find that, locally, there is uniquely defined an n-dimensional space which is either conformal or Weyl, and R^{2n} may be regarded as a product of this n-space and an n-dimensional vector space.

This suggests the construction of a Riemannian $2n$-space as a product of a non-Riemannian n-space (e.g. conformal or affine-connected) and a vector n-space; such a product we shall call a *Riemann extension*. It is possible, by means of such an extension, to relate the properties of a non-Riemannian n-space with those of certain Riemannian $2n$-spaces. An example of this is given elsewhere‡ by one of us.

The local constructions are extended to manifolds in the large, and in § 9 we show how a Riemannian structure can be given to a fibre bundle which has as base space a manifold with a given non-Riemannian structure, the tangent vector spaces being the fibres.

2. Canonical form

We consider a $2n$-dimensional Riemannian space R^{2n} $(n > 2)$ which admits a parallel field of null n-planes π. Locally there exists a canonical coordinate system§ (x^α), i.e. (x^i, ξ_i), where for convenience we have written

$$\xi_i = x^{i'} \quad (i' = i+n).$$

In terms of these coordinates the metric of R^{2n} takes the form||

$$ds^2 = g_{ij}(x,\xi)\, dx^i dx^j + 2dx^i d\xi_i, \tag{1}$$

and the fundamental tensors of R^{2n} are

$$(g_{\alpha\beta}) = \begin{pmatrix} (g_{ij}) & I \\ I & O \end{pmatrix}, \qquad (g^{\alpha\beta}) = \begin{pmatrix} O & I \\ I & -(g_{ij}) \end{pmatrix} \tag{2}$$

† A. G. Walker, *Quart. J. of Math.* (Oxford) (1) 20 (1949), 135–45; (2) 1 (1950), 69–79 and 147–52. E. M. Patterson, ibid. (2) 2 (1951), 151–8.

‡ E. M. Patterson, *J. of London Math. Soc.* (*sub judice*).

§ Greek suffixes run from 1 to $2n$, unprimed Latin suffixes from 1 to n, and primed Latin suffixes from $n+1$ to $2n$, with the convention $i' = i+n$.

|| A. G. Walker, *Quart. J. of Math.* (Oxford) (2) 1 (1950), 69–79.

Quart. J. Math. Oxford (2), 3 (1952), 19–28.

where the submatrices are of order $n \times n$; π is the plane with basis $\delta^{\alpha}_{(i')}$ $(i' = n+1,..., 2n)$.

We now suppose that π is generated by a second-order symmetric recurrent tensor,† i.e. that there exists in R^{2n} a tensor field $T_{\alpha\beta} = T_{\beta\alpha}$, of rank n, satisfying the equations

$$T_{\alpha\beta,\gamma} = T_{\alpha\beta}\nu_{\gamma} \tag{3}$$

for some vector ν_{α}, and such that $T_{\alpha\beta}\lambda^{\beta} = 0$ for every vector λ^{α} of the n-plane conjugate to π. Since π is null, it is self-conjugate, and therefore $T_{\alpha\beta}\delta^{\beta}_{(i')} = 0$, i.e. $T_{\alpha i'} = 0$. Hence

$$(T_{\alpha\beta}) = \begin{pmatrix} (T_{ij}) & O \\ O & O \end{pmatrix},$$

where $T_{ij} = T_{ji}$ and $\|T_{ij}\| \neq 0$. Calculating Christoffel symbols and substituting in (3), we find that T_{ij} are functions of the x's and ξ's satisfying

$$T_{ij.k'} = T_{ij}\nu_{k'}, \tag{4 a}$$

$$T_{ij.k} + \sum_{m} (T_{im}g_{jk.m'} + T_{jm}g_{ik.m'}) = T_{ij}\nu_k, \tag{4 b}$$

where a point denotes partial differentiation.

Equations (4 a) show that there is a scalar ψ such that $\nu_{i'} = \psi_{.i'}$. If, therefore, we consider $h_{\alpha\beta}$ in place of $T_{\alpha\beta}$, where

$$h_{\alpha\beta} = e^{-\psi}T_{\alpha\beta},$$

then $h_{\alpha\beta}$ is a symmetric recurrent tensor which generates π and is independent of the ξ's. Writing κ_{α} for the recurrence vector of $h_{\alpha\beta}$, so that $h_{\alpha\beta,\gamma} = h_{\alpha\beta}\kappa_{\gamma}$, then $\kappa_{i'} = 0$, and from equations (4 b) with h_{ij} and κ_m in place of T_{ij} and ν_m we find

$$\sum_{m} h_{km}g_{ij.m'} = -(h_{ki.j} + h_{jk.i} - h_{ij.k}) + h_{ki}\kappa_j + h_{kj}\kappa_i - h_{ij}\kappa_k. \tag{5}$$

Hence $$g_{ij.p'} = -2H^{p}_{ij} + (\delta^{p}_{i}\kappa_j + \delta^{p}_{j}\kappa_i - h^{pq}\kappa_q h_{ij}), \tag{6}$$

where (h^{ij}) is the matrix reciprocal of the $n \times n$ matrix (h_{ij}), and H^{p}_{ij} are the Christoffel symbols of the second kind formed from the h_{ij}. Differentiating (6) with respect to ξ_q we get

$$g_{ij.p'q'} = \delta^{p}_{i}\kappa_{j.q'} + \delta^{p}_{j}\kappa_{i.q'} - h^{pr}\kappa_{r.q'}h_{ij}. \tag{7}$$

Multiplying (5) throughout by $h^{ip}h^{jk}$ and summing gives

$$h^{ip}h^{jk}h_{ij.k} + \sum_{k} \tfrac{1}{2}(h^{ip}g_{ik})_{.k'} + \tfrac{1}{2}(h^{jk}g_{jk})_{.p'} = h^{pr}\kappa_r.$$

From (6) it follows that

$$\sum_{k} g_{ik.k'} = -2H^{k}_{ik} + n\kappa_i,$$

† This is equivalent to saying that there is a tensor $T_{\alpha\beta}$ such that there are n elementary divisors ρ^2 of $T_{\alpha\beta} - \rho g_{\alpha\beta}$ at each point of R^{2n}. See E. M. Patterson, *Quart. J. of Math.* (Oxford), (2) 2 (1951), 151–8.

and therefore

$$h^{pr}\kappa_r(1-\tfrac{1}{2}n) = \tfrac{1}{2}(h^{jk}g_{jk})_{.p'}+h^{ip}H^k_{ik}-h^{ip}h^{jk}h_{ij.k}.$$

Differentiating with respect to ξ_q, and assuming now that $n > 2$, we find

$$h^{pr}\kappa_{r.q'} = \frac{1}{2-n}(h^{jk}g_{jk})_{.p'q'}. \tag{8}$$

It follows from (7) and (8) that

$$\delta^p_i\,\kappa_{j.q'}+\delta^p_j\,\kappa_{i.q'} = \delta^q_i\,\kappa_{j.p'}+\delta^q_j\,\kappa_{i.p'}. \tag{9}$$

Putting $p = j$ in (9) and summing, we find

$$\kappa_{i.q'} = \frac{1}{n}\delta^q_i \sum_j \kappa_{j.j'} = \delta^q_i\,\theta, \tag{10}$$

where $\theta = \frac{1}{n}\sum_j \kappa_{j.j'}$. Hence $\kappa_{i.q'} = 0$ if $i \neq q$, and $\kappa_{i.i'}$ is the same function θ for all i. We have

$$\kappa_{i.i'j'} = \kappa_{i.j'i'} = 0 \quad (i \neq j),$$

$$\kappa_{i.i'i'} = \kappa_{j.j'i'} = \kappa_{j.i'j'} = 0 \quad (i \neq j),$$

so that θ is independent of the ξ's. Integrating (10), we find

$$\kappa_i = \theta\xi_i+\phi_i$$

for some functions ϕ_i independent of the ξ's.

From (6) and (7),

$$g_{ij.p'} = -2H^p_{ij}+(\delta^p_i\,\kappa_j+\delta^p_j\,\kappa_i-h^{pq}\kappa_q\,h_{ij}), \tag{11}$$

$$g_{ij.p'q'} = \theta(\delta^p_i\,\delta^q_j+\delta^q_i\,\delta^p_j-h^{pq}h_{ij}), \tag{12}$$

and, integrating (11), we get

$$g_{ij} = \theta(\xi_i\,\xi_j-\tfrac{1}{2}h^{pq}\xi_p\,\xi_q\,h_{ij})-2L^p_{ij}\,\xi_p+c_{ij},$$

where $$L^p_{ij} = H^p_{ij}-\tfrac{1}{2}(\delta^p_i\,\phi_j+\delta^p_j\,\phi_i-h^{pq}\phi_q\,h_{ij})$$

and the c_{ij} are arbitrary functions independent of the ξ's. Hence:

In terms of canonical coordinates (x^i, ξ_i) an R^{2n} which admits a parallel field of null n-planes generated by a second-order symmetric recurrent tensor has a metric of the form (1), *where*

$$\left.\begin{aligned} g_{ij} &= (\xi_i\,\xi_j-\tfrac{1}{2}h_{ij}\,h^{pq}\xi_p\,\xi_q)-2L^p_{ij}\,\xi_p+c_{ij} \\ L^p_{ij} &= H^p_{ij}-\tfrac{1}{2}(\delta^p_i\,\phi_j+\delta^p_j\,\phi_i-h_{ij}\,h^{pq}\phi_q) \end{aligned}\right\}, \tag{13}$$

θ, ϕ_i, c_{ij} $(= c_{ji})$ and h_{ij} $(= h_{ji})$ are functions of the x's only, $\|h_{ij}\| \neq 0$ and H^p_{ij} are Christoffel symbols formed from the h_{ij}. A recurrent tensor which generates the parallel field of null n-planes is

$$(h_{\alpha\beta}) = \begin{pmatrix} (h_{ij}) & O \\ O & O \end{pmatrix}$$

and the corresponding recurrence vector is

$$\kappa_i = \theta\xi_i + \phi_i, \qquad \kappa_{i'} = 0. \tag{14}$$

3. Transformations of the canonical form

There are three groups of transformations which leave the canonical metric given by (1) and (13) formally invariant.

[A] The coordinate transformations

$$\bar{x}^i = f^i(x), \qquad \bar{\xi}_a = \xi_i p^i_a, \qquad p^i_a = \partial x^i/\partial \bar{x}^a,$$

where the f's are any functions of the x's of non-vanishing Jacobian. For (1) and (13) to be preserved under this transformation we require

$$\bar{h}_{ab} = h_{ij} p^i_a p^j_b, \qquad \bar{c}_{ab} = c_{ij} p^i_a p^j_b, \qquad \bar{\phi}_a = \phi_i p^i_a, \qquad \bar{\theta} = \theta, \tag{15}$$

and the $\bar{H}^a_{bc}$, the Christoffel symbols formed from the $\bar{h}_{ab}$, are given by the usual connexion transformation. Thus, under all transformations of the group A, h_{ij}, c_{ij}, and ϕ_i transform as tensors in an n-space of coordinates x^i, θ is a scalar, and the L^i_{jk} are coefficients of connexion.

[B] The coordinate transformations

$$\bar{x}^i = x^i, \qquad \bar{\xi}_i = \xi_i - \eta_i,$$

where the η's are functions of the x's. This requires the substitutions

$$\left.\begin{aligned} \bar{h}_{ij} = h_{ij}, \qquad \bar{H}^i_{jk} = H^i_{jk}, \qquad \bar{\theta} = \theta, \qquad \bar{\phi}_i = \phi_i + \theta\eta_i \\ \bar{c}_{ij} = c_{ij} - 2L^p_{ij}\eta_p + \eta_{i.j} + \eta_{j.i} + \theta(\eta_i\eta_j - \tfrac{1}{2}h_{ij}h^{pq}\eta_p\eta_q) \end{aligned}\right\}. \tag{16}$$

[C] The substitution

$$\bar{h}_{ij} = e^{2\omega}h_{ij}, \qquad \bar{\phi}_i = \phi_i + 2\omega_{.i}, \qquad \bar{\theta} = \theta, \qquad \bar{c}_{ij} = c_{ij}, \tag{17}$$

where ω is any function of the x's.

We observe that, in every transformation, θ is invariant. This suggests that θ is an invariant of R^{2n}; and in fact we find, on calculating the scalar curvature R of R^{2n} from the metric (1) and (13), that

$$R = -n^2\theta. \tag{18}$$

We shall consider the two distinct cases $R \neq 0$ and $R = 0$.

4. The case $R \neq 0$

In this case a transformation of the group B can be chosen to simplify the canonical form. Taking $\eta_i = -\theta^{-1}\phi_i$ we have $\phi_i = 0$ and therefore $\bar{L}^p_{ij} = \bar{H}^p_{ij}$. Hence:

When $R \neq 0$, i.e. $\theta \neq 0$, the canonical coordinates can be chosen so that

$$g_{ij} = \theta(\xi_i\xi_j - \tfrac{1}{2}h_{ij}h^{pq}\xi_p\xi_q) - 2H^p_{ij}\xi_p + c_{ij}. \tag{19}$$

This form is preserved under every transformation of group A as

before. It is also preserved when we apply a transformation of the group C provided that this is followed by an appropriate transformation of group B. We find, for any function ω of the x's,

$$\bar{h}_{ij} = e^{2\omega}h_{ij}, \qquad \bar{x}^i = x^i, \qquad \bar{\xi}_i = \xi_i + 2\theta^{-1}\omega_{.i},$$
$$\bar{c}_{ij} = c_{ij} - 4\theta^{-1}(\omega_{.ij} - \omega_{.i}\omega_{.j} - H^p_{ij}\omega_{.p} + \tfrac{1}{2}h^{pq}\omega_{.p}\omega_{.q}h_{ij}) - {}$$
$$- \tfrac{1}{2}\theta^{-1}(\omega_{.i}\theta_{.j} + \omega_{.j}\theta_{.i}). \quad (20)$$

These are the only transformations which leave the canonical form unaltered, and it follows that, when $R \neq 0$, there is uniquely defined a conformal space CR^n, with coordinates x^i and metric $e^{2\omega}h_{ij}\,dx^i dx^j$. In addition there is in CR^n an invariant scalar θ and a tensor c_{ij}, the latter depending upon the choice of ω.

The special case when θ is constant is particularly interesting, for then the tensor c_{ij} is the sum of two tensors, one of them a conformal invariant (i.e. independent of ω) and the other a dependent tensor.

Writing H_{ij} for the Ricci tensor of the Riemannian space R^n of fundamental tensor h_{ij}, and $H = h^{ij}H_{ij}$ for the scalar curvature, then under a conformal transformation $\bar{h}_{ij} = e^{2\omega}h_{ij}$ we have†

$$\bar{H}_{ij} - \frac{\bar{H}\bar{h}_{ij}}{2(n-1)}$$
$$= H_{ij} - \frac{Hh_{ij}}{2(n-1)} + (n-2)(\omega_{.ij} - \omega_{.i}\omega_{.j} - H^p_{ij}\omega_{.p} + \tfrac{1}{2}h^{pq}\omega_{.p}\omega_{.q}h_{ij}).$$

Combining these relations with (20) and assuming that θ is constant, we get

$$\bar{c}_{ij} + \frac{4\theta^{-1}}{(n-2)}\left(\bar{H}_{ij} - \frac{\bar{H}\bar{h}_{ij}}{2(n-1)}\right) = c_{ij} + \frac{4\theta^{-1}}{(n-2)}\left(H_{ij} - \frac{Hh_{ij}}{2(n-1)}\right).$$

This shows that the tensor on the right is a conformal invariant, and we can write

$$c_{ij} = a_{ij} - \frac{4\theta^{-1}}{(n-2)}\left(H_{ij} - \frac{H}{2(n-1)}h_{ij}\right), \quad (21)$$

where a_{ij} is a conformally invariant tensor. Hence:

When R is a non-zero constant, R^{2n} defines uniquely a conformal space CR^n and a conformally invariant tensor field a_{ij} in CR^n.

From the invariance of a_{ij} we expect this tensor to be related in an intrinsic way to R^{2n}. Calculating the Ricci tensor $R_{\alpha\beta}$ for R^{2n} and remembering that $R = -n^2\theta$, we find that, if $A_{\alpha\beta}$ is the Einstein tensor

$$A_{\alpha\beta} = R_{\alpha\beta} - \frac{R}{n}g_{\alpha\beta}$$

† L. P. Eisenhart, *Riemannian Geometry* (Princeton 1926), § 28.

for R^{2n}, then
$$(A_{\alpha\beta}) = \begin{pmatrix} (A_{ij}) & O \\ O & O \end{pmatrix},$$
where
$$A_{ij} = 2H_{ij} + \tfrac{1}{2}\theta(n-2)c_{ij} + \tfrac{1}{2}\theta h^{pq}c_{pq}h_{ij},$$
i.e., from (21),
$$A_{ij} = \tfrac{1}{2}\theta\{(n-2)a_{ij} + h^{pq}a_{pq}h_{ij}\}.$$
Hence
$$a_{ij} = \frac{\theta^{-1}}{(n-2)}\left\{2A_{ij} - \frac{h^{pq}A_{pq}}{(n-1)}h_{ij}\right\}. \tag{22}$$

Thus to $A_{\alpha\beta}$ in R^{2n} there corresponds a tensor A_{ij} in CR^n, and the tensor a_{ij} is given by (22).

An immediate consequence is that, *if R^{2n} is an Einstein space, then R is constant, and $a_{ij} = 0$.* The converse is also true; *if θ is constant and $a_{ij} = 0$, then R^{2n} is an Einstein space.*

We observe that, when $\theta \neq 0$, the vector κ_α is not the gradient of a scalar, and therefore π is not generated by a tensor $T_{\alpha\beta}$ satisfying $T_{\alpha\beta,\gamma} = 0$.

5. The case $R = 0$

In this case $\theta = 0$, and the components g_{ij} are given by
$$g_{ij} = -2L^p_{ij}\xi_p + c_{ij}, \tag{23}$$
where
$$L^p_{ij} = H^p_{ij} - \tfrac{1}{2}(\delta^p_i\phi_j + \delta^p_j\phi_i - h_{ij}h^{pq}\phi_q), \tag{24}$$
H^p_{ij} being the Christoffel symbols for h_{ij}. A recurrent tensor is $h_{\alpha\beta}$ and the corresponding recurrence vector is $\kappa_i = \phi_i$, $\kappa_{i'} = 0$. Under a transformation of the group A, L^p_{ij} transform as connexion coefficients and c_{ij} as tensor components. Under a transformation of the group B, L^p_{ij} is unaltered and c_{ij} is replaced by
$$\begin{aligned}\bar{c}_{ij} &= c_{ij} - 2L^p_{ij}\eta_p + \eta_{i.j} + \eta_{j.i} \\ &= c_{ij} + \eta_{i;j} + \eta_{j;i},\end{aligned} \tag{25}$$
where the semi-colon denotes covariant differentiation with respect to the connexion L^p_{ij}.

The connexion is restricted because of (24), and we see that the n-space is in fact a Weyl† space W^n with fundamental tensor h_{ij} and fundamental vector ϕ_i. The group of substitutions C is the usual group associated with a Weyl space. The tensor c_{ij} is dependent on the choice of fundamental tensor in W^n. Hence:

When $R = 0$, there is uniquely defined in R^{2n} an n-dimensional Weyl space W^n, with coordinates x^i, connexion L^p_{ij}, fundamental tensor h_{ij} and fundamental vector ϕ_i.

† L. P. Eisenhart, *Non-Riemannian Geometry*, American Math. Soc. Colloquium publication, vol. 8, § 30.

A particular case arises when the recurrent tensor which generates π can be chosen so that its covariant derivative vanishes. This occurs only when κ_α is a gradient in R^{2n}, i.e. ϕ_i is a gradient $\phi_{,i}$ in W^n. In this case ϕ_i can be transformed away by a transformation of the group C; the L^p_{ij} then become Christoffel symbols for the fundamental tensor. Hence:

When the recurrent tensor has a zero covariant derivative, $R = 0$, and the Weyl space has the connexion of a Riemannian space.

6. Riemann extensions

The foregoing results suggest ways of constructing Riemannian $2n$-spaces out of certain kinds of non-Riemannian n-spaces. For a construction of this kind to be interesting the $2n$-space should determine uniquely the n-space out of which it is constructed. Such a $2n$-space we shall call a *Riemann extension* of the n-space.

In §§ 7 and 8 we define Riemann extensions locally. In § 9 we consider the problem in the large.

7. Riemann extensions of conformal spaces

Let CR^n be a conformal n-space, with coordinates x^i and fundamental tensor h_{ij} (determinate to within a scalar multiplier). Let θ be a nonzero constant, and $\mathbf{a} = (a_{ij})$ a symmetric tensor field in CR^n. We define the Riemann extension $R^{2n}(CR^n, \mathbf{a}, \theta)$ to be the space with coordinates x^i, ξ_i and metric

$$ds^2 = g_{ij}\,dx^i dx^j + 2dx^i d\xi_i$$

where

$$\left.\begin{aligned} g_{ij} &= \theta(\xi_i \xi_j - \tfrac{1}{2}h_{ij}h^{pq}\xi_p \xi_q) - 2H^p_{ij}\xi_p + c_{ij} \\ c_{ij} &= a_{ij} - \frac{4\theta^{-1}}{(n-2)}\left(H_{ij} - \frac{1}{2(n-1)}h_{ij}\right) \end{aligned}\right\}, \tag{26}$$

H^p_{ij} are the Christoffel symbols for h_{ij}, H_{ij} the Ricci tensor, and H the scalar curvature.

From the results of §§ 2 and 3 we know that this $2n$-space determines the coordinate system (x^i) to within allowable transformations, and that in any such system the tensor h_{ij} is determinate except for an arbitrary scalar multiplier; the constant θ and tensor a_{ij} are unique. Thus R^{2n} defines CR^n, $\mathbf{a}$, and θ uniquely, as required. The significance of the constant θ is that it determines the metric scale of R^{2n}. This can be seen from the relation $R = -n^2\theta$.

In particular, we can always choose the Riemann extension of a conformal space to be an Einstein space. This is done by taking $\mathbf{a} = 0$.

We thus have a method of constructing non-trivial Einstein spaces of even dimensionality.

It should be noted that the R^{2n} constructed in this way is severely restricted. It admits a recurrent tensor which generates a parallel field of null n-planes, and it has constant scalar curvature. It can easily be verified, for example, that R^{2n} cannot be conformally flat, and hence, in particular, cannot be a space of constant curvature.

8. Riemann extensions of affine-connected spaces

Let A^n be an affine-connected space, with coordinates x^i and symmetric connexion L^p_{ij}. We define the Riemann extension $R^{2n}(A^n)$ to be the space of coordinates x^i, ξ_i and metric

$$ds^2 = g_{ij}\,dx^i dx^j + 2dx^i d\xi_i,$$

where

$$g_{ij} = -2L^p_{ij}\,\xi_p. \tag{27}$$

Although in § 4 we were concerned with the restricted case when A^n is a Weyl space, most of the arguments of that section can be applied here with the L's as general symmetric connexion coefficients. Under any coordinate transformation leaving the metric form invariant, L^p_{ij} transform as connexion coefficients, and A^n is therefore determined uniquely by R^{2n}.

This extension can be generalized to include a tensor c_{ij} by taking

$$g_{ij} = -2L^p_{ij}\,\xi_p + c_{ij} \tag{28}$$

instead of (27). This may sometimes be useful but it is not entirely satisfactory since c_{ij} is not uniquely determined by R^{2n}; it follows from (25) that a tensor of the form $\eta_{i;j}+\eta_{j;i}$ can always be added to c_{ij} by means of the transformation $\bar{x}^i = x^i$, $\bar{\xi}_i = \xi_i - \eta_i$.

From results for a Weyl space we expect the scalar curvature of R^{2n} to be zero for any A^n. In fact we find that the Ricci tensor of the R^{2n} with metric (1) and (28) is given by

$$(R_{\alpha\beta}) = \begin{pmatrix} (B_{ij}+B_{ji}) & O \\ O & O \end{pmatrix},$$

where B_{ij} is the contracted curvature tensor for A^n. It follows at once that $R = g^{\alpha\beta}R_{\alpha\beta} = 0$.

9. Riemann extensions in the large

Suppose now that we have an n-dimensional topological manifold M^n. Steenrod† has shown how a manifold can be constructed as a fibre bundle with M^n as base space, the fibre being the vector space associated with

† N. E. Steenrod, *Annals of Math.* 43 (1942), 116–31.

tensors of a specified type. In particular, there is a $2n$-dimensional manifold M^{2n} constructed in this way by taking the fibres to be the tangent n-spaces of covariant vectors. If x^i are allowable coordinates in a neighbourhood $C(x)$ of M^n, and if ξ_i is a covariant vector, then (x^i, ξ_i) are the coordinates of a point of M^{2n} over the point x^i of M^n.

When M^n has either a conformal metric or an affine connexion, we can apply our local Riemann extensions to each coordinate neighbourhood of M^n and assign a Riemannian structure to the fibre bundle M^{2n}. In this way we get a Riemann extension in the large with the property that the base space M^n and its differential structure are uniquely determinate when M^{2n} is known.

Consider firstly the case when M^n has a conformal metric. This means that, in each C.N. (coordinate neighbourhood) there is a conformal metric tensor field k_{ij}, i.e. at each point of M^n we have tensor component ratios $k_{ij}:k_{pq}:\ldots$, where $\|k_{ij}\| \neq 0$, in each allowable coordinate system. For our construction we must prove that there is a non-singular tensor field $\mathbf{h}$ in M^n such that at each point the tensor components of the field have the given ratios. To do this we use the fact that every manifold† is Riemannian, i.e. that there exists a non-singular second-order symmetric tensor field $\mathbf{m}$. Let m_{ij} be the components of $\mathbf{m}$ in a C.N., and define

$$h_{ij} = \frac{\|m_{pq}\|^{1/n}}{\|k_{pq}\|^{1/n}} k_{ij}.$$

Then h_{ij} are the components of a tensor and are uniquely defined in each allowable coordinate system and each C.N. We thus have a tensor field h_{ij} as required, such that $h_{ij} \propto k_{ij}$ at every point.

For generality, we choose a non-zero constant θ, and suppose that we are given a second-order symmetric tensor field $\mathbf{a}$ over M^n. If the components of $\mathbf{a}$ in a C.N. $C(x)$ of M^n are a_{ij}, then at the point (x^i, ξ_i) of the fibre bundle M^{2n} we take the fundamental tensor $g_{\alpha\beta}$ to be given by the metric (26). Since transformations which leave the metric formally invariant are those relating allowable coordinate systems in $C(x)$, it follows that we have defined a fundamental tensor field $\mathbf{g}$, i.e. a Riemannian metric, in M^{2n}. If $\mathbf{a}$ is chosen to be zero, then M^{2n} is an Einstein manifold.

Conversely, taking M^{2n} as given, then we know that it contains a parallel field of null n-planes such that these are tangents to a system of submanifolds; these are the fibres of M^{2n} and they determine the base space. In a C.N. of M^{2n} the coordinates x^i, ξ_i of the canonical form

† N. E. Steenrod, loc. cit.

are found as before; the x's are coordinates in a neighbourhood of the base space M^n, and the canonical form gives θ, a_{ij}, and the conformal metric tensor h_{ij} of M^n. We have thus recovered all the original elements.

The extension of an affine-connected manifold is defined similarly. We have in each C.N. a symmetric affine connexion L^p_{ij} which transforms in the usual way when the allowable coordinates are transformed. At the point (x^i, ξ_i) of the fibre bundle M^{2n} we take the fundamental tensor $\mathfrak{g}$ of M^{2n} to be given by the metric (27). The fibres, base space, and connexion can be recovered as before when M^{2n} is given.

A NOTE ON TWO OF RAMANUJAN'S FORMULAE

By W. N. BAILEY (*London*)

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The formula*

$$\sum_{n=0}^{\infty} p(5n+4)x^n = 5\prod_{n=1}^{\infty} \frac{(1-x^{5n})^5}{(1-x^n)^6}, \tag{1}$$

where $p(n)$ is the number of partitions of n, was given by Ramanujan,† and proofs were given by Darling and Mordell.

I have lately seen a copy of an unpublished manuscript by Ramanujan, kindly lent to me by Prof. Watson, in which a different proof is given which is not without interest. Ramanujan states the formula

$$\frac{x\{(1-x^5)(1-x^{10})(1-x^{15})...\}^5}{(1-x)(1-x^2)(1-x^3)...} = \frac{x}{(1-x)^2}-\frac{x^2}{(1-x^2)^2}-\frac{x^3}{(1-x^3)^2}+$$

$$+\frac{x^4}{(1-x^4)^2}+\frac{x^6}{(1-x^6)^2}-\frac{x^7}{(1-x^7)^2}-.... \tag{2}$$

He then equates terms in the expansions of these two sides which involve powers of x which are multiples of 5, and obtains the formula

$$\prod_{n=1}^{\infty}(1-x^{5n})^5\sum_{n=0}^{\infty} p(5n+4)x^{5n+5} = \frac{5x^5}{(1-x^5)^2}-\frac{5x^{10}}{(1-x^{10})^2}-....$$

On changing x^5 into x, this gives

$$\prod_{n=1}^{\infty}(1-x^{n})^5\sum_{n=0}^{\infty} p(5n+4)x^{n+1} = 5\left[\frac{x}{(1-x)^2}-\frac{x^2}{(1-x^2)^2}-...\right],$$

where the series in square brackets is the same as that on the right of (2), and, using (2) to express the right-hand side as a product, we immediately obtain (1).

The only gap in this argument is the proof of (2), but this formula

* In his obituary notice of Ramanujan, Hardy remarks that, if he had to select one (beautiful) formula from all Ramanujan's work, he would agree with Major MacMahon in selecting (1).

† See Ramanujan's *Collected Papers*, 213. References to the papers by Darling and Mordell are given on page 343.

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follows easily from the known sum of a well-poised basic bilateral hypergeometric series.* The right-hand side of (2) is

$$\sum_{n=0}^{\infty}\left[\frac{x^{5n+1}}{(1-x^{5n+1})^2}-\frac{x^{5n+2}}{(1-x^{5n+2})^2}-\frac{x^{5n+3}}{(1-x^{5n+3})^2}+\frac{x^{5n+4}}{(1-x^{5n+4})^2}\right]$$

$$=\sum_{n=-\infty}^{\infty}\left[\frac{x^{5n+1}}{(1-x^{5n+1})^2}-\frac{x^{5n+3}}{(1-x^{5n+3})^2}\right]$$

$$=\sum_{n=-\infty}^{\infty}\frac{x^{5n+1}(1-x^2)(1+x^{5n+2})(1-x^{5n+2})}{(1-x^{5n+1})^2(1-x^{5n+3})^2}$$

$$=\frac{x(1-x^2)(1-x^4)}{(1-x)^2(1-x^3)^2}\,{}_6\Psi_6\left[\begin{matrix}x^7,\ -x^7,\ x,\ x,\ x^3,\ x^3;\\ x^2,\ -x^2,\ x^8,\ x^8,\ x^6,\ x^6\end{matrix}\ x^5\right]$$

$$=\frac{x(1-x^2)(1-x^4)}{(1-x)^2(1-x^3)^2}\times$$

$$\times\prod_{n=1}^{\infty}\left[\frac{(1-x^{5n+4})(1-x^{5n+2})(1-x^{5n})^4(1-x^{5n-2})(1-x^{5n-4})}{(1-x^{5n-1})^2(1-x^{5n-3})^2(1-x^{5n+3})^2(1-x^{5n+1})^2}\right]$$

$$=x\prod_{n=1}^{\infty}\frac{(1-x^{5n})^5}{1-x^n},$$

where $q = x^5$ in the bilateral series. This completes the proof.

The other formula which I wish to consider is

$$\frac{\{(1-x)(1-x^2)(1-x^3)...\}^5}{(1-x^5)(1-x^{10})(1-x^{15})...}$$
$$=1-5\left(\frac{x}{1-x}-\frac{2x^2}{1-x^2}-\frac{3x^3}{1-x^3}+\frac{4x^4}{1-x^4}+\frac{6x^6}{1-x^6}-...\right), \tag{3}$$

which was stated by Ramanujan in his unpublished manuscript, but was also given elsewhere. It can be expressed in the form that, if

$$f(x)=x^{\frac{1}{5}}\frac{(1-x)(1-x^4)(1-x^6)(1-x^9)...}{(1-x^2)(1-x^3)(1-x^7)(1-x^8)...},$$

then
$$\frac{df}{f\,dx}=\frac{1}{5x}\prod_{n=1}^{\infty}\frac{(1-x^n)^5}{1-x^{5n}}.$$

A very complicated proof of this result was given by Darling, and a short but more sophisticated proof was given by Mordell.† I find

* W. N. Bailey, *Quart. J. of Math.* (Oxford), 7 (1936) 113.

† In the papers already referred to.

that (3) can also be deduced from the same sum of a bilateral series. In fact, from this sum we have

$$
{}_6\Psi_6\begin{bmatrix} q\sqrt{a},\ -q\sqrt{a},\ \omega\sqrt{a},\ \omega^2\sqrt{a},\ \omega^3\sqrt{a},\ \omega^4\sqrt{a}; \\ \sqrt{a},\ -\sqrt{a},\ \omega^4q\sqrt{a},\ \omega^3q\sqrt{a},\ \omega^2q\sqrt{a},\ \omega q\sqrt{a} \end{bmatrix} q
$$

$$
= \prod_{n=1}^{\infty} \frac{(1-q^n)(1-aq^n)(1-q^n/a)(1-q^{5n})(1-q^n\sqrt{a})(1-q^n/\sqrt{a})}{(1-q^{5n}a^{\frac{5}{2}})(1-q^{5n}/a^{\frac{5}{2}})},
$$

where $\omega = \exp(2\pi i/5)$. Now, if $a \to 1$, the product on the right becomes

$$
\prod_{n=1}^{\infty} \frac{(1-q^n)^5}{1-q^{5n}}.
$$

The series on the left is

$$
1+\frac{1-a^{\frac{5}{2}}}{(1-a)(1-a^{\frac{1}{2}})} \sum_{-\infty}^{\infty}{}' \frac{(1-aq^{2n})(1-a^{\frac{1}{2}}q^n)q^n}{1-a^{\frac{5}{2}}q^{5n}}
$$

$$
= 1+\frac{1-a^{\frac{5}{2}}}{(1-a)(1-a^{\frac{1}{2}})} \sum_{n=1}^{\infty} \left[\frac{(1-aq^{2n})(1-a^{\frac{1}{2}}q^n)q^n}{1-a^{\frac{5}{2}}q^{5n}} - \frac{(q^{2n}-a)(q^n-a^{\frac{1}{2}})q^n}{a^{\frac{5}{2}}-q^{5n}}\right].
$$

The expression in square brackets has the numerator

$$
-a^{\frac{3}{2}}q^n(1-a)+aq^{2n}(1-a^2)+a^{\frac{1}{2}}q^{3n}(1-a^3)-q^{4n}(1-a^4)-
$$

$$
-q^{6n}(1-a^4)+a^{\frac{1}{2}}q^{7n}(1-a^3)+aq^{8n}(1-a^2)-a^{\frac{3}{2}}q^{9n}(1-a),
$$

and so the limit of the series, when $a \to 1$, is

$$
1-5\sum_{n=1}^{\infty}\left[\frac{q^n-2q^{2n}-3q^{3n}+4q^{4n}+4q^{6n}-3q^{7n}-2q^{8n}+q^{9n}}{(1-q^{5n})^2}\right].
$$

Now $$\sum_{n=1}^{\infty}\frac{q^{an}}{(1-q^{5n})^2} = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty}(m+1)q^{(5m+a)n} = \sum_{m=0}^{\infty}\frac{(m+1)q^{5m+a}}{1-q^{5m+a}},$$

and so we have

$$
\prod_{n=1}^{\infty}\frac{(1-q^n)^5}{1-q^{5n}} = 1-5\sum_{m=0}^{\infty}\left[\frac{(m+1)q^{5m+1}}{1-q^{5m+1}} - \frac{2(m+1)q^{5m+2}}{1-q^{5m+2}} - \right.
$$

$$
-\frac{3(m+1)q^{5m+3}}{1-q^{5m+3}} + \frac{4(m+1)q^{5m+4}}{1-q^{5m+4}} + \frac{4mq^{5m+1}}{1-q^{5m+1}} -
$$

$$
\left. -\frac{3mq^{5m+2}}{1-q^{5m+2}} - \frac{2mq^{5m+3}}{1-q^{5m+3}} + \frac{mq^{5m+4}}{1-q^{5m+4}}\right]
$$

$$
= 1-5\sum_{m=0}^{\infty}\left[\frac{(5m+1)q^{5m+1}}{1-q^{5m+1}} - \frac{(5m+2)q^{5m+2}}{1-q^{5m+2}} - \right.
$$

$$
\left. -\frac{(5m+3)q^{5m+3}}{1-q^{5m+3}} + \frac{(5m+4)q^{5m+4}}{1-q^{5m+4}}\right].
$$

This completes the proof of (3).

LINEAR FORMS ASSOCIATED WITH AN ALGEBRAIC NUMBER-FIELD

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1. Let K be an algebraic number-field of degree n, and let $K_1,..., K_n$ be the algebraically conjugate fields of which K is one. We shall suppose that $K_1,..., K_r$ are real fields, that $K_{r+1},..., K_n$ are complex, and that the fields K_{r+j} and K_{r+s+j} are conjugate complex for $j = 1,..., s$, where $r+2s = n$. Let $\omega_{11},..., \omega_{1n}$ be a basis for the algebraic integers of K_1, and $\omega_{p1},..., \omega_{pn}$ the corresponding basis for K_p $(p = 2,..., n)$. The linear forms

$$\left.\begin{array}{l} \xi_1 = \omega_{11}x_1+...+\omega_{1n}x_n \\ \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ \xi_n = \omega_{n1}x_1+...+\omega_{nn}x_n \end{array}\right\} \tag{1}$$

represent the general algebraic integer of K_1 and its algebraic conjugates, when $x_1,..., x_n$ take all integral† values. The determinant of the forms (1) is $\sqrt{d}$, where d is the discriminant of K.

In a recent paper‡ L. E. Clarke proved that, if $\alpha_1,..., \alpha_n$ is any set of numbers of the same type as the field K (that is, $\alpha_1,..., \alpha_r$ real and $\bar{\alpha}_{r+j} = \alpha_{r+s+j}$ for $j = 1,..., s$), then there exist integers $x_1,..., x_n$ such that

$$|(\xi_1-\alpha_1)...(\xi_n-\alpha_n)| < C_n|d|^{\frac{1}{2}n}, \tag{2}$$

where C_n is a constant depending only on n. This is best possible (as far as the exponent of $|d|$ is concerned) when K is a complex quadratic field, but otherwise not. When K is a totally real field, the result is valid with the exponent $\frac{1}{2}$ in the place of $\frac{1}{2}n$, this being a consequence of Tschebotareff's theorem§ on the product of n non-homogeneous linear forms. Clarke showed that the exponent $\frac{1}{2}n$ in (2) could be improved to

$$\tfrac{1}{2}n-(n-2)/(n-1)$$

when n is a prime.

In this paper, I obtain a substantial improvement on (2), given by the following theorem.

† It will be convenient to restrict the word *integer*, without the qualification *algebraic*, to mean *rational* integer.

‡ *Quart. J. of Math.* (Oxford) (2) 2 (1951), 308–15.

§ See, for example, Hardy and Wright, *Introduction to the Theory of Numbers*, (2nd ed., 1945), 396.

THEOREM 1. *Let $\alpha_1,\ldots,\alpha_n$ be any set of numbers of the same type as the field K. Then there exist integers $x_1,\ldots,x_n$ such that*

$$|(\xi_1-\alpha_1)\ldots(\xi_n-\alpha_n)| < C_n|d|^{n/2(n-s)}, \tag{3}$$

where C_n is a constant depending only on n.

If the field is totally real, so that $s = 0$, the exponent is $\frac{1}{2}$, and the result is then of the same form as that mentioned above. Apart from this, the simplest case with which to illustrate the theorem is that of a cubic field of negative discriminant. Here $r = s = 1$, and the exponent is $\frac{3}{4}$. It is doubtful whether this is the best possible or not.

As regards results in the opposite direction, I have proved† that for any cubic field of negative discriminant there exist α's such that the product on the left of (3) is always greater than a constant multiple of $|d|^{\frac{1}{3}}$. Clarke has proved that a better result holds for cubic fields of a special kind (those generated by $(l^3+1)^{\frac{1}{3}}$, where l satisfies certain conditions), namely with $|d|^{\frac{3}{8}}$ in place of $|d|^{\frac{1}{3}}$.

The general idea of the proof is to choose $r+s$ positive parameters $\lambda_1,\ldots,\lambda_{r+s}$ so that the quadratic form

$$\lambda_1\xi_1^2+\ldots+\lambda_r\xi_r^2+2\lambda_{r+1}|\xi_{r+1}|^2+\ldots+2\lambda_{r+s}|\xi_{r+s}|^2 \tag{4}$$

in $x_1,\ldots,x_n$ has a special property, namely that of its n successive minima (in the sense of Minkowski), the last $r+s$ are approximately equal. It is very probable that the parameters can be so chosen as to make these minima exactly equal, but it seems difficult to prove such a result except when the number of parameters is small.‡ For our purpose, it suffices to obtain equality within a constant factor, and this we do by modifying a method which was devised by Siegel§ for the case $s = 0$. A note on the scope of this method is added in § 8.

The same method proves also the following result:

THEOREM 2. *For any ideal $\mathfrak{a}$ in K and any number α (not necessarily integral) in K, there exists an algebraic integer η in K such that $\eta \equiv \alpha \pmod{\mathfrak{a}}$ and*

$$|N\eta| < C_n(N\mathfrak{a})|d|^{n/2(n-s)}. \tag{5}$$

The device used by Clarke to improve on the exponent in (2) when n is a prime can be combined with the present method, provided that $s > 1$, and we then obtain:

† *Acta Math.* 84 (1950), 159–79.

‡ See Dyson's proof [*Annals of Math.* 49 (1948), 82–109] in the case $r = 4$, $s = 0$.

§ See H. Davenport, *Acta Arithmetica*, 2 (1937), 262–5.

THEOREM 3. *If $s > 1$ and n is a prime, the exponent of $|d|$ in* (3) *and* (5) *can be replaced by*

$$\frac{n}{2(n-s)}-\frac{s-1}{(n-1)(n-s)}.$$

2. Most of our arguments are independent of the particular arithmetical character of the linear forms $\xi_1,..., \xi_n$, and, until we come to the proof of Theorem 1 in § 5, we shall suppose only that the forms are real and complex as already stated, and have a determinant Δ which is not zero.

Let $q(\mathbf{x}) = q(x_1,..., x_n)$ denote the quadratic form

$$\left.\begin{aligned} q(x_1,...,x_n) &= \xi_1^2+...+\xi_r^2+|\xi_{r+1}|^2+...+|\xi_n|^2 \\ &= \xi_1^2+...+\xi_r^2+2|\xi_{r+1}|^2+...+2|\xi_{r+s}|^2 \end{aligned}\right\}. \tag{6}$$

This is a positive definite form, and its determinant D is given by

$$D = |\Delta|^2. \tag{7}$$

The successive minima $\tau_1,..., \tau_n$ of q are defined as follows: τ_1 is the minimum of $q(\mathbf{x})$ for all integral $\mathbf{x}$ other than 0, attained say for $\mathbf{x}_1$; τ_2 is the minimum of $q(\mathbf{x})$ for all integral $\mathbf{x}$ that are not proportional to $\mathbf{x}_1$, attained say for $\mathbf{x}_2$; τ_3 is the minimum of $q(\mathbf{x})$ for all $\mathbf{x}$ that are not linearly dependent on $\mathbf{x}_1$ and $\mathbf{x}_2$, and so on. Plainly

$$0 < \tau_1 \leqslant \tau_2 \leqslant ... \leqslant \tau_n.$$

LEMMA 1. *We have*

$$D \leqslant \tau_1 \tau_2 ... \tau_n \leqslant c_n D, \tag{8}$$

where c_n depends only on n.

Proof. These inequalities are due to Minkowski,† but as the proof is not difficult we give it here. We can suppose, after applying an integral unimodular substitution to the variables $x_1,..., x_n$, that the points $\mathbf{x}_1,..., \mathbf{x}_n$ at which the successive minima are attained have the special form

$$\mathbf{x}_1 = (x_{11}, 0, 0,...),\ \mathbf{x}_2 = (x_{21}, x_{22}, 0,...),....$$

By the process of completing the square, we can express q as

$$q = \kappa_1(x_1+\theta_{12}x_2+...)^2+\kappa_2(x_2+\theta_{23}x_3+...)^2+...+\kappa_n x_n^2,$$

where $\kappa_1,..., \kappa_n$ are positive numbers whose product is D. Since

$$\tau_1 = q(\mathbf{x}_1) \geqslant \kappa_1,\ \tau_2 = q(\mathbf{x}_2) \geqslant \kappa_2,...,$$

we have
$$\tau_1 \tau_2 ... \tau_n \geqslant \kappa_1 \kappa_2 ... \kappa_n = D.$$

† *Geometrie der Zahlen*, § 51.

To prove the second inequality, we consider the form

$$q^*(\mathbf{x}) = \frac{\kappa_1}{\tau_1}(x_1+\theta_{12}x_2+\ldots)^2+\frac{\kappa_2}{\tau_2}(x_2+\theta_{23}x_3+\ldots)^2+\ldots+\frac{\kappa_n}{\tau_n}x_n^2,$$

of determinant $D(\tau_1\tau_2\ldots\tau_n)^{-1}$. If $q(\mathbf{x}) \geqslant \tau_n$, then $q^*(\mathbf{x}) \geqslant 1$. If

$$\tau_{n-1} \leqslant q(\mathbf{x}) < \tau_n,$$

then $\mathbf{x}$ must be linearly dependent on $\mathbf{x}_1,\ldots, \mathbf{x}_{n-1}$, so that $x_n = 0$. In this case we again get $q^*(\mathbf{x}) \geqslant 1$. The argument applies generally, and shows that $q^*(\mathbf{x}) \geqslant 1$ for all integral $\mathbf{x}$ other than 0. Now define $1/c_n$ to be the lower bound of the determinants of all quadratic forms in n variables whose minimum is 1. It is well known that this lower bound exists, and various estimates have been given for it. Since the minimum of $q^*(\mathbf{x})$ is at least 1, we have

$$D(\tau_1\tau_2\ldots\tau_n)^{-1} \geqslant 1/c_n.$$

This completes the proof of (8).

3. We now introduce the adjoint linear forms to the forms (1), and write them as

$$\left.\begin{array}{l} \Xi_1 = \Omega_{11}X_1+\ldots+\Omega_{1n}X_n \\ \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \\ \Xi_n = \Omega_{n1}X_1+\ldots+\Omega_{nn}X_n \end{array}\right\}. \tag{9}$$

Here Ω_{hk} is the cofactor of ω_{hk} in the coefficient-matrix of (1), divided by the determinant Δ. It is easily verified that $\Xi_1,\ldots, \Xi_r$ are real, and that Ξ_{r+j} is complex conjugate to Ξ_{r+s+j} for $j = 1,\ldots, s$. We have

$$\xi_1\Xi_1+\ldots+\xi_n\Xi_n = x_1X_1+\ldots+x_nX_n \tag{10}$$

identically in the $2n$ variables.

Let $$Q(X_1,\ldots, X_n) = \Xi_1^2+\ldots+\Xi_r^2+2|\Xi_{r+1}|^2+\ldots+2|\Xi_{r+s}|^2. \tag{11}$$

Then Q is the adjoint form to q, and its determinant is $D^{-1} = |\Delta|^{-2}$. Let $T_1,\ldots, T_n$ be the successive minima of the form Q.

LEMMA 2. *We have* $$1 \leqslant \tau_h T_{n+1-h} \leqslant c_n^2 \tag{12}$$
for $h = 1, 2,\ldots, n$.

Proof. This result (in a more general form) is due to Mahler,† but again the proof is quite simple. Let $\mathbf{x}_1,\ldots, \mathbf{x}_n$ be, as before, the points at which q attains its successive minima, and $\mathbf{X}_1,\ldots, \mathbf{X}_n$ those at which Q attains its successive minima. Since both these sets of points are linearly independent, it follows (as is most easily seen by geometrical considerations) that not all the scalar products $\mathbf{x}_p\mathbf{X}_q$ can be zero, where

† *Časopis Mat. a Fys.* 68 (1939), 93–102.

$p = 1,\dots, h$ and $q = 1,\dots, n+1-h$. If this scalar product is not zero, its absolute value is at least one, and so by (10)

$$|\xi_1 \Xi_1+\dots+\xi_n \Xi_n| \geqslant 1$$

for the values of the linear forms corresponding to $\mathbf{x}_p$ and $\mathbf{X}_q$. Now

$$\begin{aligned}|\xi_1 \Xi_1+\dots+\xi_n \Xi_n| &\leqslant \{\xi_1^2+\dots+|\xi_n|^2\}^{\frac12}\{\Xi_1^2+\dots+|\Xi_n|^2\}^{\frac12} \\ &= \tau_p T_q \\ &\leqslant \tau_h T_{n+1-h}.\end{aligned}$$

This proves the left-hand half of (12). The right-hand half follows on multiplying together (8) and the analogous inequality

$$D^{-1} \leqslant T_1 T_2 \dots T_n \leqslant c_n D^{-1}, \tag{13}$$

and using the fact that $\tau_k T_{n+1-k} \geqslant 1$ for every k other than h.

4. We now modify q by introducing positive parameters $\lambda_1,\dots, \lambda_{r+s}$ satisfying

$$\lambda_1 \dots \lambda_r (\lambda_{r+1} \dots \lambda_{r+s})^2 = 1. \tag{14}$$

We put

$$q_\lambda(\mathbf{x}) = \lambda_1 \xi_1^2+\dots+\lambda_r \xi_r^2+2\lambda_{r+1}|\xi_{r+1}|^2+\dots+2\lambda_{r+s}|\xi_{r+s}|^2. \tag{15}$$

In view of (14), the determinant of $q_\lambda(\mathbf{x})$ is still D, and the adjoint form is

$$Q_\lambda(\mathbf{X}) = \lambda_1^{-1}\Xi_1^2+\dots+\lambda_r^{-1}\Xi_r^2+2\lambda_{r+1}^{-1}|\Xi_{r+1}|^2+\dots+2\lambda_{r+s}^{-1}|\Xi_{r+s}|^2. \tag{16}$$

We denote by $\tau_1(\lambda),\dots, \tau_n(\lambda)$ and $T_1(\lambda),\dots, T_n(\lambda)$ the successive minima of the forms q_λ and Q_λ. These satisfy (8), (12), and (13) for any choice of the λ's.

Lemma 3. *There exist positive numbers $\lambda_1,\dots, \lambda_{r+s}$ satisfying* (14), *such that*

$$T_1(\lambda) \geqslant (n!)^{-1}\{(T_1 \dots T_s)^2 T_{s+1} \dots T_{s+r}\}^{1/n}. \tag{17}$$

Proof. We express the complex forms among the forms (9) in terms of real forms by putting $\Xi_j \sqrt{2} = Z_j+iZ_{j+s}$ for $j = r+1,\dots, r+s$. Then $\Xi_1,\dots, \Xi_r, Z_{r+1},\dots, Z_n$ are n real linear forms in $X_1,\dots, X_n$ of determinant $|\Delta|^{-1}$, and

$$Q(\mathbf{X}) = \Xi_1^2+\dots+\Xi_r^2+Z_{r+1}^2+\dots+Z_n^2.$$

We denote by $L_1,\dots, L_n$ a certain permutation of these n real forms, obtained as follows. The inequality $Q(\mathbf{X}) < T_n$ implies a linear relation between $X_1,\dots, X_n$, which is equivalent to a certain linear relation

$$A_1 L_1+\dots+A_n L_n = 0,$$

where the coefficients are real numbers, not all zero. We select L_n so that A_n is numerically the greatest coefficient. The inequality

$$Q(\mathbf{X}) < T_{n-1}$$

implies a further linear relation, which we can take in the form

$$B_1 L_1 + \ldots + B_{n-1} L_{n-1} = 0.$$

We select L_{n-1} to be the form whose coefficient is numerically greatest, and so on. This defines $L_1, \ldots, L_n$ as a certain permutation of $\Xi_1, \ldots, Z_n$.

The inequality $Q(\mathbf{X}) < T_n$ implies that

$$|L_n| \leqslant |L_1| + \ldots + |L_{n-1}|,$$

whence
$$L_n^2 \leqslant (n-1)(L_1^2 + \ldots + L_{n-1}^2),$$

i.e.
$$L_1^2 + \ldots + L_{n-1}^2 \geqslant \frac{1}{n}(L_1^2 + \ldots + L_n^2) = \frac{1}{n} Q(\mathbf{X}).$$

The inequality $Q(\mathbf{X}) < T_{n-1}$ implies in addition that

$$L_{n-1}^2 \leqslant (n-2)(L_1^2 + \ldots + L_{n-2}^2),$$

whence
$$L_1^2 + \ldots + L_{n-2}^2 \geqslant \frac{1}{n(n-1)} Q(\mathbf{X}).$$

Generally, the inequality $Q(\mathbf{X}) < T_{h+1}$ implies that

$$L_1^2 + \ldots + L_h^2 \geqslant \frac{1}{n(n-1)\ldots(h+1)} Q(\mathbf{X}).$$

It follows that for any integral $\mathbf{X} \neq 0$ there is a value of h for which

$$L_1^2 + \ldots + L_h^2 \geqslant (n!)^{-1} T_h.$$

As a consequence, we have

$$\Phi(\mathbf{X}) = T_1^{-1} L_1^2 + \ldots + T_n^{-1} L_n^2 \geqslant (n!)^{-1} \tag{18}$$

for all integral $\mathbf{X} \neq 0$.

Put $\Xi_1 = L_{m_1}, \ldots, Z_n = L_{m_n}$, where $m_1, \ldots, m_n$ is a permutation of $1, \ldots, n$. The form $\Phi(\mathbf{X})$ is not of the type (16), since we do not know that the coefficients of Z_j and Z_{j+s} are equal. We modify Φ to meet this requirement by replacing the corresponding coefficients by whichever is the greater, for each pair of forms Z_j, Z_{j+s}. Let

$$\left.\begin{aligned} U_1 = T_{m_1}, \ldots, U_r = T_{m_r} \\ U_{r+1} = \min(T_{m_{r+1}}, T_{m_{r+s+1}}), \ldots, U_{r+s} = \min(T_{m_{r+s}}, T_{m_n}) \end{aligned}\right\}. \tag{19}$$

The modified quadratic form is

$$\left.\begin{aligned} \Psi(\mathbf{X}) &= \frac{\Xi_1^2}{U_1} + \ldots + \frac{\Xi_r^2}{U_r} + \frac{Z_{r+1}^2 + Z_{r+s+1}^2}{U_{r+1}} + \ldots + \frac{Z_{r+s}^2 + Z_n^2}{U_{r+s}} \\ &= \frac{\Xi_1^2}{U_1} + \ldots + \frac{\Xi_r^2}{U_r} + \frac{2|\Xi_{r+1}|^2}{U_{r+1}} + \ldots + \frac{2|\Xi_{r+s}|^2}{U_{r+s}} \end{aligned}\right\}. \tag{20}$$

Since Ψ is obtained from Φ by increasing certain coefficients, we have $\Psi(\mathbf{X}) \geqslant (n!)^{-1}$ for all integral $\mathbf{X} \neq 0$.

The form (20) is of the type (16), except that its coefficients do not satisfy (14). To ensure this, we take

$$\lambda_j = U_j\{U_1 \dots U_r (U_{r+1} \dots U_{r+s})^2\}^{-1/n}.$$

With these values for the parameters, the form (16) has

$$T_1(\lambda) \geqslant (n!)^{-1}\{U_1 \dots U_r (U_{r+1} \dots U_{r+s})^2\}^{1/n}.$$

It remains to find a lower bound for the product on the right, where $U_1, \dots, U_{r+s}$ are given by (19). We have to consider what permutation $m_1, \dots, m_n$ of $1, \dots, n$ makes this product least. In that permutation, all the numbers $1, 2, \dots, s$ must occur among $m_{r+1}, \dots, m_n$. For, if one of them did not, say $s' = m_{r'}$, where $s' \leqslant s$ and $r' \leqslant r$, there would be one pair m_{r+j}, m_{r+s+j} which had both its members greater than s, and we could diminish the product by interchanging $m_{r'}$ with the lesser of m_{r+j}, m_{r+s+j}. Again, exactly one of the numbers $1, 2, \dots, s$ must occur in each pair m_{r+j}, m_{r+s+j}, for, if two occurred in the same pair, we could diminish the product by interchanging the greater of them with any one of $m_1, \dots, m_r$, since these numbers are all greater than s. Thus the minimum value of the product arises when

$$U_{r+1} \dots U_{r+s} = T_1 \dots T_s$$

and

$$U_1 \dots U_r = T_{s+1} \dots T_{s+r}.$$

This gives

$$T_1(\lambda) \geqslant (n!)^{-1}\{(T_1 \dots T_s)^2 T_{s+1} \dots T_{s+r}\}^{1/n},$$

as stated in the enunciation.

LEMMA 4. *There exist positive numbers $\lambda_1, \dots, \lambda_{r+s}$ satisfying* (14) *such that*

$$\frac{\tau_n(\lambda)}{\tau_{s+1}(\lambda)} < (n!)^{n+1} c_n^2. \tag{21}$$

Proof. It will suffice to prove that we can choose the λ's so that

$$\frac{T_{r+s}(\lambda)}{T_1(\lambda)} < (n!)^{n+1}, \tag{22}$$

since then (21) follows by (12).

We have already seen in the preceding lemma that we can choose parameters λ for which

$$T_1(\lambda) \geqslant (n!)^{-1}\{(T_1 \dots T_s)^2 T_{s+1} \dots T_{s+r}\}^{1/n} \geqslant (n!)^{-1}\{T_1^{n-1} T_{r+s}\}^{1/n}.$$

If $T_{r+s} < (n!)^{n+1} T_1$, the desired result (22) is satisfied with all the λ's equal to 1. If this is not the case, we have $T_1(\lambda) \geqslant (n!)^{1/n} T_1$. Now repeat the argument on the linear forms $\lambda_1^{-\frac{1}{2}} \Xi_1, \dots, \lambda_{r+s}^{-\frac{1}{2}} \Xi_{r+s}$. If the desired result (22) is not satisfied for the parameters λ, we obtain a set of parameters μ for which $T_1(\mu) \geqslant (n!)^{1/n} T_1(\lambda)$. This continues until a set of parameters

is reached for which (22) holds. The process cannot continue indefinitely, since $T_1(\lambda)$ cannot exceed $(c_n D^{-1})^{1/n}$, by (13).

5. Proof of Theorem 1

We choose parameters $\lambda_1,\ldots,\lambda_{r+s}$ as in Lemma 4. Let $\xi_1^{(j)},\ldots,\xi_n^{(j)}$ for $j = 1,\ldots,n$ be the values of the linear forms (1) corresponding to the point $\mathbf{x}_j$ at which $q_\lambda(\mathbf{x})$ assumes its jth minimum value $\tau_j(\lambda)$. These n sets of values of the n forms are linearly independent, by the definition of the successive minima. Hence any set of numbers $\alpha_1,\ldots,\alpha_n$ of the same type as K is representable as

$$\alpha_p = \theta_1 \xi_p^{(1)} + \ldots + \theta_n \xi_p^{(n)} \quad (p = 1,\ldots,n),$$

where the numbers θ_j are real. By taking $\xi_1,\ldots,\xi_n$ to be a suitable linear combination of the values $\xi_1^{(j)},\ldots,\xi_n^{(j)}$ for $j = 1,\ldots,n$, with integral multipliers, we can ensure that

$$\xi_p - \alpha_p = \phi_1 \xi_p^{(1)} + \ldots + \phi_p \xi_p^{(n)} \quad (p = 1,\ldots,n),$$

where $-\frac{1}{2} < \phi_j \leqslant \frac{1}{2}$ for $j = 1,\ldots,n$.

We now have

$$|\xi_p - \alpha_p|^2 \leqslant \tfrac{1}{4}n\big(|\xi_p^{(1)}|^2 + \ldots + |\xi_p^{(n)}|^2\big).$$

Since $\lambda_p |\xi_p^{(j)}|^2 \leqslant \tau_j(\lambda)$, it follows that

$$\lambda_p |\xi_p - \alpha_p|^2 \leqslant \tfrac{1}{4}n(\tau_1 + \ldots + \tau_n) \leqslant \tfrac{1}{4}n^2 \tau_n,$$

where τ_j is now an abbreviation for $\tau_j(\lambda)$. Thus

$$\lambda_1 |\xi_1 - \alpha_1|^2 + \ldots + \lambda_r |\xi_r - \alpha_r|^2 + 2\lambda_{r+1} |\xi_{r+1} - \alpha_{r+1}|^2 + \ldots +$$
$$+ 2\lambda_{r+s} |\xi_{r+s} - \alpha_{r+s}|^2 \leqslant \tfrac{1}{4}n^3 \tau_n. \tag{23}$$

By Lemmas 4 and 1 (where now $D = |\Delta|^2 = |d|$) we have

$$\tau_1 \ldots \tau_s \big(\tau_n (n!)^{-n-1} c_n^{-2}\big)^{n-s} \leqslant c_n |d|,$$

whence

$$\tau_n \leqslant k_n |d|^{1/(n-s)} (\tau_1 \ldots \tau_s)^{-1/(n-s)}, \tag{24}$$

where k_n depends only on n.

Any value of the form $q_\lambda(\mathbf{x})$ arising from integral $\mathbf{x} \neq 0$ must be at least equal to n, since

$$q_\lambda(\mathbf{x}) = \lambda_1 \xi_1^2 + \ldots + 2\lambda_{r+s} |\xi_{r+s}|^2 \geqslant n |\xi_1 \ldots \xi_n|^{2/n} \geqslant n,$$

on using the inequality of the arithmetic and geometric means, and (14), and the fact that the norm of a non-zero algebraic integer of K is a non-zero integer. Hence $\tau_1 \geqslant n$, and (24) gives

$$\tau_n < k_n |d|^{1/(n-s)}. \tag{25}$$

Substituting in (23) and again using the inequality of the arithmetic and geometric means and (14), we obtain

$$|(\xi_1-\alpha_1)...(\xi_n-\alpha_n)| < C_n|d|^{n/2(n-s)},$$

which is (3).

6. Proof of Theorem 2

If $\eta \equiv \alpha \pmod{\mathfrak{a}}$, then $\eta = \alpha+\xi$, where ξ is an algebraic integer in the ideal $\mathfrak{a}$. We apply the preceding method to the linear forms $\xi_1,..., \xi_n$ which represent the general algebraic integer ξ of the ideal $\mathfrak{a}$ and its algebraic conjugates. The determinant of these linear forms is

$$\Delta = (N\mathfrak{a})\sqrt{d},$$

and their product is an integral multiple of $N\mathfrak{a}$ for any integral $\mathbf{x}$. The proof proceeds as before, and we obtain (24) with $|d|$ replaced by $(N\mathfrak{a})^2|d|$. In place of $\tau_1 \geqslant n$ we now have

$$\tau_1 \geqslant n(N\mathfrak{a})^{2/n},$$

by a similar use of the inequality of the arithmetic and geometric means. Thus

$$\tau_n < k_n\{(N\mathfrak{a})^2|d|\}^{1/(n-s)}(N\mathfrak{a})^{-2s/n(n-s)},$$

and the final result is of the form stated in (5), with $\eta = \xi+\alpha$.

7. Proof of Theorem 3

It was proved by Clarke (loc. cit.) that, if n is a prime, then

$$\tau_2 \geqslant k_n'|d|^{2/n(n-1)}.$$

If $s > 1$, we can use this in (24) to improve on (25), obtaining

$$\tau_n < k_n''(|d|^{1-2(s-1)/n(n-1)})^{1/(n-s)}.$$

This leads to the result given in Theorem 3.

8. The method of choosing the parameters $\lambda_1,..., \lambda_{r+s}$ which was used in §§ 3, 4 applies equally well in a more general situation. Suppose we have n real linear forms $Z_1,..., Z_n$ in n variables $x_1,..., x_n$ and suppose them grouped together in any way. Let there be k groups, with n_1, $n_2,..., n_k$ forms in the various groups, where $n_1+...+n_k = n$. Let S_ν denote the sum of the squares of the linear forms in the νth group. We can associate with the grouped linear forms the quadratic form

$$q_\lambda(\mathbf{x}) = \lambda_1 S_1+...+\lambda_k S_k,$$

where $\lambda_1,..., \lambda_k$ are positive parameters. If we impose on them the condition

$$\lambda_1^{n_1}\lambda_2^{n_2}...\lambda_k^{n_k} = 1,$$

then the determinant of the quadratic form is independent of the choice of the parameters. Let the n successive minima of $q_\lambda(\mathbf{x})$ be $\tau_1(\lambda),..., \tau_n(\lambda)$.

Then the same argument as before shows that we can choose $\lambda_1,\ldots,\lambda_k$ so that the first k minima have bounded ratios. Or, alternatively, by considering the adjoint forms, we can choose the parameters so that the last k minima have bounded ratios.

Whether any result follows for the non-homogeneous problem which corresponds to the grouped linear forms, depends on whether any lower bound is known for $\tau_1(\lambda)$. The following problem, though somewhat artificial, is one where such a lower bound is obvious, as it was in the application made in § 5. Let $Y_1,\ldots, Y_m$ be linear forms in $x_1,\ldots, x_{2m}$ with coefficients which are algebraic integers in a real quadratic field, and let $Y'_1,\ldots, Y'_m$ be the algebraically conjugate forms. Let Δ be the determinant of the $2m$ forms $Y_1,\ldots, Y'_m$, and suppose $\Delta \neq 0$. Then

$$(Y_1^2+\ldots+Y_m^2)(Y_1'^2+\ldots+Y_m'^2) \geqslant 1 \tag{26}$$

for all integers $x_1,\ldots, x_{2m}$ which are not all zero. We can choose λ so that the quadratic form

$$\lambda(Y_1^2+\ldots+Y_m^2)+\lambda^{-1}(Y_1'^2+\ldots+Y_m'^2)$$

has its last two minima in bounded ratio. Also $\tau_1(\lambda) \geqslant 2$, by (26). It follows that for any $2m$ real numbers $\alpha_1,\ldots, \alpha_m, \alpha'_1,\ldots, \alpha'_m$, there are integers $x_1,\ldots, x_{2m}$ such that

$$\{(Y_1-\alpha_1)^2+\ldots+(Y_m-\alpha_m)^2\}\{(Y'_1-\alpha'_1)^2+\ldots+(Y'_m-\alpha'_m)^2\} < C_m\Delta^2,$$

where C_m depends only on m. A crude argument, without using any parameter, would give Δ^4 instead of Δ^2.

VARIANTS OF A CLASSICAL ISOPERIMETRIC PROBLEM

By A. S. BESICOVITCH (*Cambridge*)

[Received 2 February 1951]

VARIANT II†

PROBLEM. *Among the convex sets with a given length of boundary and a given circumradius find one of maximum area.*

It will be shown that this problem is equivalent to the Favard problem: *What is the smallest circle on which every closed convex curve of length l and area S can be placed?* The answer is that it is the circle circumscribed about the symmetrical lens belonging to the above class of sets.‡ My proof is independent of that of M. J. Favard.

The circumcircle of a convex curve Γ has either (i) a pair A, B of diametrically opposite points in common with the curve, or (ii) has three points A, B, C in common, such that every arc AB, BC, CA of the circle is less than 180°.

Let Γ_1 be, in the case (i), the lens with vertices at A and B formed by two circular arcs of lengths equal to the lengths of the arcs into which Γ is divided by the points A, B, and in the case (ii), the curve formed by the circular arcs subtending the pairs of points A, B; B, C; C, A and of lengths equal to the lengths of the arcs AB, BC, CA of Γ respectively. In either case $\Lambda\Gamma_1 = \Lambda\Gamma$ and either Γ coincides with Γ_1 or $|\Gamma| < |\Gamma_1|$.§ Thus we can confine our problem only to the class of curves like Γ_1, from which it follows that a curve Γ_0 of maximum area exists and that it is either a lens or a circular triangle.

All the arcs (2 or 3) forming Γ_0 are of the same radius. For suppose that Γ_0 is a circular triangle ABC (Fig. 1) and that the arcs AC and CB have different radii. Take on them arcs $\alpha\beta$ and $\alpha'\beta'$ subtended by equal chords and draw the arcs $\alpha\delta\beta$ and $\alpha'\delta'\beta'$ of radii equal to those of the arcs CB and AC respectively, the curve $\Gamma' = A\alpha\delta\beta C\alpha'\delta'\beta' BA$ has the same length of boundary, the same area, and the same circumradius

† For Variant 1 see A. S. Besicovitch, 'A variant of a classical isoperimetric problem', *Quart. J. of Math.* (Oxford) 20 (1949), 84–94.

‡ T. Bonnesen, *Les Problèmes des isoperimètres et des isépiphanes* (Collection Borel), 167.

§ $\Lambda\Gamma$ and $|\Gamma|$ stand for the length and the area of Γ respectively.

as Γ_0. Thus it is also a curve of maximum area and yet its arcs AC, CB are not circular. This contradiction proves the statement for the case when Γ_0 is a triangle, and a similar argument is valid for the case when Γ_0 is a lens.

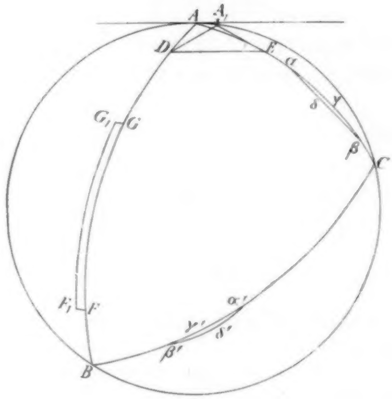

FIG. 1.

If Γ_0 is a circular triangle, then all its sides are equal. For suppose that Γ_0 is the circular triangle ABC and that $AB \neq AC$. Take a small chord DE $(=\epsilon)$ parallel to the tangent at A and the point A_1 on the circle equidistant from D and E, and form the linear triangle A_1DE. We obviously have

$$0 \neq \eta = \Lambda ADEA - \Lambda A_1DEA_1 = \omega(\epsilon)$$

and $$|ADEA| - |A_1DEA_1| = O(\epsilon^3),$$

where the notation $f(\epsilon) = \omega(\epsilon)$ indicates existence of positive numbers k, K such that, for small ϵ, $k\epsilon < |f(\epsilon)| < K\epsilon$.

Consider now the figure $\Gamma' = A_1DGG_1F_1FBCEA_1$, where F, G is a pair of fixed points (independent of ϵ) on AB, $FF_1 = GG_1 = \frac{1}{2}\eta$, FF_1 and GG_1 are parallel, and the arc F_1G_1 is equal to the arc FG.

We have $$\Lambda\Gamma' = \Lambda\Gamma_0, \qquad 0 < |\Gamma'| - |\Gamma_0| = \omega(\epsilon),$$

i.e. $$|\Gamma'| > |\Gamma_0|.$$

The last inequality would be in contradiction with the definition of Γ_0 if Γ' were convex. It is not, but its convex hull Γ'' is, and it satisfies the inequalities

$$\Lambda\Gamma'' < \Lambda\Gamma_0, \qquad |\Gamma''| > |\Gamma_0|.$$

This being impossible our statement is proved.

Thus Γ_0 *is either a symmetrical lens or a regular circular triangle* (that is a triangle formed by circular arcs of the same radius subtending vertices of a regular linear triangle).

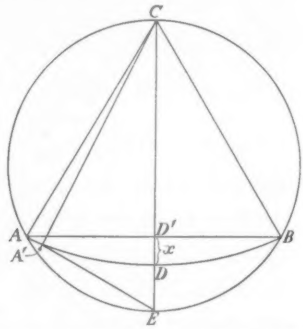

FIG. 2. (A' is not on the arc ADB.)

I shall prove that Γ_0 is a lens by showing that a regular circular triangle of the same length of boundary has not the maximum area in the class of figures we consider.

Let A, B, C be the vertices of a regular circular triangle inscribed in a unit circle (Fig. 2) (only the side ADB joining the vertices A and B is shown on the figure), CE be a diameter, $DD' = x$ $(0 < x < \frac{1}{2})$. Consider also the triangle $CA'E$, where $CA' = CA$ and $A'E = AD$. We have

$$\cos CAD = \frac{\frac{3}{2}-3x}{2\sqrt{\{3(\frac{3}{4}+x^2)\}}}, \qquad \cos CA'E = \frac{x^2-\frac{1}{4}}{2\sqrt{\{3(\frac{3}{4}+x^2)\}}},$$

$$\cos CAD+\cos CA'E = \frac{x^2-3x+\frac{5}{4}}{2\sqrt{\{3(\frac{3}{4}+x^2)\}}} > 0.$$

Since $\cos CA'E < 0$, it follows that $\sin CAD < \sin CA'E$, whence $|\Delta EA'C| > |\Delta DAC|$. Subtend the sides CA', $A'E$ by the arcs equal to those subtending the sides CA, AD and reflect the figure in CE; we shall obtain a circular quadrangle Γ' inscribed in our circle whose boundary is equal to that of the circular triangle, but whose area is greater. This completes the solution.

I shall now show that (i) the above result follows from the Favard theorem and (ii) conversely.

(i) Let Γ be a closed convex curve different from a symmetrical lens such that $\Lambda\Gamma = l$ and $|\Gamma| = S$ and U' a symmetrical lens of the same length of boundary and of the same area. By the Favard theorem the

circumradius R of Γ is less than the circumradius R' of U'. Let now U be a symmetrical lens, whose axis is $2R$ and length of the boundary l. It is easy to see that $|U| > |U'|$† and hence

$$|\Gamma| < |U|.$$

(ii) Let now a convex curve Γ and a symmetrical lens U have the same length of boundary and the same circumradius. By my theorem $|\Gamma| < |U|$.

Let now U' be another symmetrical lens of the same length of boundary and of the same area as Γ. Then U' has a longer axis, and a longer circumradius than U, which proves the Favard theorem.

VARIANT III

PROBLEM. *Find a convex set Γ with boundary of length l contained in a bounded closed convex set P, and having the largest possible area.*

The existence of such a set follows from general principles. We shall first consider some sets of points belonging to P. Let r_0 be the radius of a largest circle included in P (inscribed circle). For $0 < r \leqslant r_0$ denote by $C(r)$ the sum point-set of all the circles of radius r included in P. We have

(i) $\Lambda C(r_0) \geqslant 2\pi r_0$;

(ii) $\Lambda C(r)$ is a decreasing function of r;

(iii) $C(r'') \subset C(r')$ for any $0 < r' < r'' \leqslant r_0$;

(iv) $C(r)$ is a closed convex set;

(v) through every point M of $\mathrm{Fr}C(r)$‡ passes one and only one circle of $C(r)$ and it touches $\mathrm{Fr}\,C(r)$. Thus $\mathrm{Fr}\,C(r)$ has a tangent at every point.

(vi) $\mathrm{Fr}\,C(r)-\mathrm{Fr}\,P$ is a set $\sum\gamma$ of open arcs. Every arc γ is a circular arc of radius r. For let M be a point of γ and M' another point of γ near M. Let further c, c' be the circles of $C(r)$ through M, M' respectively. If M' is sufficiently near M, one of the two circles of radius r through M and M' belongs to $C(r)$ and thus, by (v), the circles c and c' coincide, from which (vi) follows.

† The area of a circular segment whose arc is of fixed length, but whose chord is variable, attains its maximum when the segment becomes a semicircle. When the chord increases from its value for the semicircle the area of the segment decreases. Hence the area of a symmetrical (convex) lens of a given length of boundary decreases as the axis increases.

‡ Fr for frontier.

(vii) If there is only one inscribed circle in P, then $C(r_0)$ is this circle. If there are more than one inscribed circles, then let c, c' be two of them and AA', BB' the two common outer tangents and A, A', B, B' the points of contact. From the convexity of P it follows that AA' and BB' belong to P; moreover they belong to the boundary of P, since otherwise it would be possible to inscribe in P a circle of radius greater than r_0.

It is easy to see that all the circles of $C(r_0)$ lie between the two tangents possibly prolonged one or both ways. Thus $C(r_0)$ consists of the two extreme circles of this strip and of the part of the strip between them.

We now turn to our problem.

LEMMA 1. $\mathrm{Fr}\,\Gamma - \mathrm{Fr}\,P$ *is a set of circular arcs.*

For let AMB be an arc of $\mathrm{Fr}\,\Gamma - \mathrm{Fr}\,P$. The lemma follows from the fact that for a fixed length of AMB the area between the chord AB and the arc AMB reaches its maximum value when AMB is a circular arc.

LEMMA 2. *If r is the radius of one of the arcs of* $\mathrm{Fr}\,\Gamma - \mathrm{Fr}\,P$, *then at no point of* $\mathrm{Fr}\,\Gamma$ *is the upper curvature greater than* $1/r$.

For suppose the lemma not true. Then we can find an arc A_1MA_2 of $\mathrm{Fr}\,\Gamma$ as small as we please such that the circular arc A_1NA_2 of radius r on the same side of A_1A_2 as A_1MA_2 belongs to Γ^o. Let then $A'A''$ be an arc of $\mathrm{Fr}\,\Gamma - \mathrm{Fr}\,P$ of radius r. If the arc A_1MA_2 is sufficiently small, then the figure $A_1MA_2NA_1$ can be translated into a position $A_1'M'A_2'N'A_1' \subset P^o$ where $A_1'N'A_2' = A_1NA_2$ lies on $A'A''$. Denote by Γ' the figure obtained from Γ by this translation. Then Γ' is also of the maximum area, which is impossible since the arc $A'A_1'M'A_2'A''$ of $\mathrm{Fr}\,\Gamma' - \mathrm{Fr}\,P$ is not a circular arc.

COROLLARY 1. *If $l < \mathrm{Fr}\,P$, then at every point of Γ a tangent exists, and all the arcs of* $\mathrm{Fr}\,\Gamma - \mathrm{Fr}\,P$ *are of the same radius.*

LEMMA 3. *Every arc of* $\mathrm{Fr}\,\Gamma - \mathrm{Fr}\,P$ *is less than* $180°$ *when their common radius r is less than r_0.*

Let AB be an arc of $\mathrm{Fr}\,\Gamma - \mathrm{Fr}\,P$.

(i) Suppose first that AB is an arc of $180°$. The tangents at A and B to Γ, which are also the tangents to P, are parallel and distant $2r$ from each other. The set P is included between the tangents, which is impossible since the diameter of the inscribed circle exceeds $2r$.

(ii) Suppose now that AB is more than 180°. Let O and M be the centre and the mid-point of the arc AB. A small displacement of Γ in the direction of OM will bring Γ into P^o. Since Γ in its new position is also a curve of maximum area and does not contain any points of $\operatorname{Fr} P$, we conclude that Γ is a circle of radius r, which contradicts the condition that $l = \Lambda \operatorname{Fr} \Gamma > 2\pi r_0$. Thus the lemma is proved.

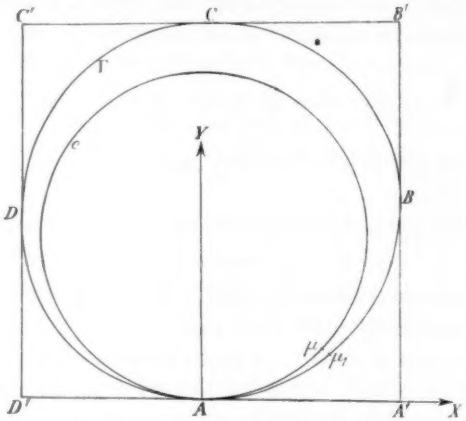

FIG. 3. (For μ, μ_1 read M, M_1.)

LEMMA 4. *If at all points of a closed convex curve Γ the upper curvature does not exceed r^{-1}, then any circle of radius r having contact with Γ and lying together with Γ on the same side of the common tangent is included in Γ.*

Let c be a circle and A the point of contact (Fig. 3). Circumscribe about Γ the rectangle $A'B'C'D'$ one of whose sides is the tangent at A, the other sides being tangents at B, C, D. Take AX, AY for coordinate axes and let $\phi(s)$ and $\phi_1(s)$ be the angles between the x-axis and the tangents at M and M_1 respectively, where M and M_1 are the points of c and Γ such that $\Lambda\smile AM = \Lambda\smile AM_1 = s$. We have

$$\phi(s) \geqslant \phi_1(s), \qquad s_1 < s_2 \to \phi(s_2) - \phi(s_1) \geqslant \phi_1(s_2) - \phi_1(s_1),$$
$$\Lambda\smile AB \geqslant \tfrac{1}{2}\pi r.$$

The coordinates of M and M_1 are

$$x = \int_0^s \cos\phi(s)\,ds, \qquad y = \int_0^s \sin\phi(s)\,ds;$$

$$x_1 = \int_0^s \cos\phi_1(s)\,ds, \qquad y_1 = \int_0^s \sin\phi_1(s)\,ds.$$

Hence, for $0 < s \leqslant \frac{1}{2}\pi r$, $x \leqslant x_1$ and $y \geqslant y_1$, and thus M_1 is outside or on the boundary of c. If $\Lambda \smile AB > \frac{1}{2}\pi r$, then, for $\frac{1}{2}\pi r \leqslant s \leqslant \Lambda \smile AB$, x_1 and y_1 are increasing functions, and thus the whole arc AB (and similarly AD) is outside or on the boundary of c. Since, for $s = \frac{1}{2}\pi r$, $x = r$ and therefore $x_1 \geqslant r$, we have $AA' \geqslant r$. Similarly $A'B, BB', \ldots, D'A$ are all not less than r. Taking the circles c', c'' of radius r, touching Γ inwardly at B and D respectively, we shall find that $\smile BC$ and $\smile DC$ are outside or on the boundaries of c', c'' respectively, from which the lemma follows.

COROLLARY 2. *If the radius of the arcs of* $\mathrm{Fr}\,\Gamma - \mathrm{Fr}\,P$ *is* r, *then* $r \leqslant r_0$ *and* $\Gamma \subset C(r)$.

THEOREM. (i) *If* $l = \Lambda\Gamma > \Lambda C(r_0)$, *then* Γ *coincides with* $C(r)$ *for the value of* r *for which* $l = \Lambda C(r)$.

(ii) *If* $\Lambda C(r_0) \geqslant l > 2\pi r_0$, *then* Γ *either coincides with* $C(r_0)$ (*when* $l = \Lambda C(r_0)$) *or is a part of* $C(r_0)$ *between a pair of its circles.*

(i) In this case the radius r of arcs of $\mathrm{Fr}\,\Gamma - \mathrm{Fr}\,P$ is less than r_0, for otherwise we should have $\Gamma \subset C(r_0)$ and thus $\Lambda\Gamma \leqslant \Lambda C(r_0)$.

Take any domain of $P^o - \Gamma$. It is bounded by an arc of $\mathrm{Fr}\,P$ and by an arc γ of $\mathrm{Fr}\,\Gamma - \mathrm{Fr}\,P$, which by Lemma 3 is less than 180°. This domain lies in the finite domain between γ and the tangents to γ at its end points, and it is clear that no circle of $C(r)$ can have points in common with this domain.

(ii) In this case $r = r_0$. For, if $r < r_0$, we should conclude, as above, that Γ coincides with $C(r)$ and that $l = \Lambda\Gamma = \Lambda C(r) > \Lambda C(r_0)$. The result now follows from the fact that $\Gamma \subset C(r_0)$.

Example. P is a regular polygon and r_0 the radius of the incircle. For $r < r_0$, $C(r)$ is P minus the parts of P cut off by the circles of radius r inscribed in every angle of P.

VARIANT IV

PROBLEM. *Find a convex set* Γ *with boundary of length* l *containing a convex set* P, *and having the largest possible area.*

Let r_0 be the radius of the circumcircle of P (there is only one such circle). The problem arises only for the case when $l < 2\pi r_0$. Its solution is analogous to that of the problem of Variant III.

LEMMA 1. $\mathrm{Fr}\,\Gamma - \mathrm{Fr}\,P$ *is a set of circular arcs.*

LEMMA 2. *If* r *is the radius of an arc of* $\mathrm{Fr}\,\Gamma - \mathrm{Fr}\,P$, *then the lower curvature at every point of* $\mathrm{Fr}\,\Gamma$ *is greater than or equal to* r^{-1} (*so that* Γ *is a convex curve*).

COROLLARY 1. *All the arcs of* $\mathrm{Fr}\,\Gamma - \mathrm{Fr}\,P$ *are of the same radius.*

LEMMA 3. *If* $r > r_0$, *every arc of* $\mathrm{Fr}\,\Gamma - \mathrm{Fr}\,P$ *is less than* $180°$.

LEMMA 4. *Any circle of radius r having inward contact with* Γ *contains the whole of* Γ.

COROLLARY 2. *We have always* $r \geqslant r_0$.

For $r \geqslant r_0$, denote by $D(r)$ the product set of all circles containing P. $D(r_0)$ is simply the circumcircle. From Lemma 4 it follows that Γ is the intersection of all the circles of radius r having inward contact with Γ. *Hence* $D(r) \subset \Gamma$.

If $l < 2\pi r_0$ then the radius r of arcs of $\mathrm{Fr}\,\Gamma - \mathrm{Fr}\,P$ exceeds r_0, for otherwise Γ would contain $D(r_0)$, that is the circumcircle, and we should have $\Lambda\Gamma \geqslant 2\pi r_0$.

THEOREM. *If* $l = \Lambda\Gamma < 2\pi r_0$, *then* Γ *coincides with* $D(r)$ *for the value of* r *for which* $l = \Lambda D(r)$.

Example. If P is a regular polygon, then $D(r)$ is the convex figure obtained from P by joining every pair of its consecutive vertices by an arc of radius r.

THE PLETHYSM OF S-FUNCTIONS

By E. M. IBRAHIM (*Bangor*)

[Received 20 February 1951]

THE importance of the evaluation of the plethysm of S-functions $\{\lambda\} \otimes \{\mu\}$ has been shown by Littlewood.* Known methods of calculation are, however, laborious. The method which involves least calculation is the 'third method'† of Littlewood. It has unfortunately the disadvantage that ambiguity arises at various stages in the calculation, and recourse is needed to some method so as to indicate the correct choice.

During the calculation of tables showing the expansion of $\{\lambda\} \otimes \{\mu\}$ for all partitions up to a total degree of 18 in the resultant, it has been found that the use of the following theorems, taken in conjunction with Littlewood's third method, is quite sufficient to determine all expansions without ambiguity. These tables have been deposited with the Royal Society Mathematical Tables Committee. Copies of these tables are available on request and application should be made to the Royal Society.

Use is made of the concept of *principal part*.‡ The terms in the product of two tensors of type $\{\lambda_1,..., \lambda_r\}$ and $\{\mu_1,..., \mu_r\}$ respectively may correspond to any or all of the terms which appear in the product of these S-functions. The term of type $\{\lambda_1+\mu_1,..., \lambda_r+\mu_r\}$, however, is the only term which is never zero. It is called the *principal part of the product*. A concomitant is only regarded as reducible if it is the principal part of a product of concomitants of lower degrees or is a linear combination of principal parts.

If the number of parts in one partition is less than in the other partition, zero parts are added to make them equal. This applies in every case throughout the paper.

For algebraic forms which are in Clebsch form,§ the principal part is the only non-zero term in the product.

THEOREM I. *The principal parts of the products of individual terms in the expansion of*

$$[\{\lambda_1,..., \lambda_r\} \otimes \{n\}][\{\mu_1,..., \mu_r\} \otimes \{n\}]$$

* Littlewood (2), (3).

† Ibid. (3), 349.

‡ Ibid. (2), 398.

§ Ibid. (3), 324.

appear as terms in the expansion of

$$\{\lambda_1+\mu_1,\ldots, \lambda_r+\mu_r\} \otimes \{n\}.$$

Let g and h be symbolic expressions for quantics in Clebsch form which are of types $\{\lambda_1,\ldots, \lambda_r\}$ and $\{\mu_1,\ldots, \mu_r\}$ respectively. If the same symbols are used in the two expressions, then $f = gh$ may be regarded as the symbolic expression for a form of type $\{\lambda_1+\mu_1,\ldots, \lambda_r+\mu_r\}$.

Let ϕ and ψ be symbolic expressions of degree n in g and n in h. If the same symbols are used in each expression, then $\phi\psi$ will give the symbolic expression for a concomitant of degree n in f. The existence of this concomitant proves the theorem.

The theorem does not imply that frequency of occurrence of a partition in $\{\lambda_1+\mu_1,\ldots, \lambda_r+\mu_r\} \otimes \{n\}$ is at least as great as the number of ways in which it appears as a principal part of products of terms in $(\{\lambda\} \otimes \{n\})(\{\mu\} \otimes \{n\})$. Thus

$$(\{2\} \otimes \{2\})(\{2\} \otimes \{2\}) = (\{4\}+\{2^2\})(\{4\}+\{2^2\})$$

gives principal parts $\{8\}+2\{62\}+\{4^2\}$. But in $\{4\} \otimes \{2\} = \{8\}+\{62\}+\{4^2\}$ the coefficient of $\{62\}$ is unity. Each product $\{4\}\{2^2\}$ corresponds to the same concomitant. A coefficient greater than one can be assumed when these coefficients appear in the individual forms of $\{\lambda\} \otimes \{n\}$ or $\{\mu\} \otimes \{n\}$. Thus $\{5\} \otimes \{4\}$ includes $3\{12.62\}$ and $\{1\} \otimes \{4\} = \{4\}$, hence in $\{6\} \otimes \{4\}$ the coefficient of $\{16.62\}$ is at least 3. The corresponding concomitants are obviously linearly independent.

Analysis of the corresponding concomitants may indicate that a coefficient even higher than this might be inferred. But this involves careful analysis of the structure of the concomitants and generally the above principle is found, in practice, to be sufficient. The same principle applies to subsequent theorems.

THEOREM II. *The principal parts of products of terms of the expressions*

$$[\{\lambda_1,\ldots, \lambda_r\} \otimes \{1^n\}][\{\mu_1,\ldots, \mu_r\} \otimes \{1^n\}]$$

appear as terms in the expansion of

$$\{\lambda_1+\mu_1,\ldots, \lambda_r+\mu_r\} \otimes \{n\}.$$

Let there be n ground forms of type $\{\lambda_1,\ldots, \lambda_r\}$ and let the ith be expressed symbolically in terms of the symbols $\alpha_i, \alpha_i', \alpha_i'',\ldots$.

Let there be another set of n ground forms of type $\{\mu_1,\ldots, \mu_r\}$ and express the ith symbolically in terms of the same symbols $\alpha_i, \alpha_i', \alpha_i'',\ldots$. The product of the symbolic expressions for the ith ground form in each set will be of degree $(\lambda_1+\mu_1)$ in α_i, $(\lambda_2+\mu_2)$ in α_i', etc., and may be

interpreted as a ground form of type $\{\lambda_1+\mu_1,..., \lambda_r+\mu_r\}$. Let ϕ be a symbolic expression giving an alternating concomitant linear in each of the first set of ground forms and let ψ be a symbolic expression giving an alternating concomitant linear in each of the second set of ground forms. Then the product $\phi\psi$ may be interpreted as a concomitant of n ground forms of type $\{\lambda_1+\mu_1,..., \lambda_r+\mu_r\}$. But the effect of interchanging two of these ground forms will be to change the sign of both ϕ and ψ. Thus $\phi\psi$ will be unaltered. It is therefore a symmetric concomitant. This proves the theorem.

THEOREM III. *The principal parts of products of terms from the expressions*

$$[\{\lambda_1,..., \lambda_r\} \otimes \{n\}][\{\mu_1,..., \mu_r\} \otimes \{1^n\}]$$

appear as terms in the expansion of

$$\{\lambda_1+\mu_1,..., \lambda_r+\mu_r\} \otimes \{1^n\}.$$

Follow the same proof as in Theorem II but let ϕ this time be a symbolic expression giving a symmetric concomitant linear in each of the first set of ground forms and ψ a symbolic expression giving an alternating concomitant linear in each of the second set of ground forms.

Then the product $\phi\psi$ may be interpreted as a concomitant of n ground forms of type $\{\lambda_1+\mu_1,..., \lambda_r+\mu_r\}$. But the effect of interchanging two of these ground forms will be to change only the sign of ψ. Thus $\phi\psi$ will be altered. It is therefore an alternating concomitant. This proves the theorem.

THEOREM IV. *The principal parts of products of terms from the expressions*

$$[\{\lambda\} \otimes \{\mu_1,..., \mu_r\}][\{\lambda\} \otimes \{\nu_1,..., \nu_r\}]$$

appear as terms in the expansion of

$$\{\lambda\} \otimes \{\mu_1+\nu_1,..., \mu_r+\nu_r\}.$$

Let ϕ be a concomitant of class* $\{\mu\} \equiv \{\mu_1,..., \mu_r\}$ of r ground forms $f_1, f_2,..., f_r$ each of type $\{\lambda\}$, and let the degree of ϕ be μ_1 in the coefficients of f_1, μ_2 in the coefficients of f_2, etc. Similarly let ψ be a concomitant of class $\nu \equiv \{\nu_1,..., \nu_r\}$ of the same r ground forms and of degree ν_1 in the first, ν_2 in the second, etc. It will now be shown that the product $\phi\psi$ must necessarily be of class $\{\mu_1+\nu_1,..., \mu_r+\nu_r\}$.

Since the class of ϕ is $\{\mu\}$ and the class of ψ is $\{\nu\}$, therefore the class of $\phi\psi$ must correspond to a term in the product $\{\mu\}\{\nu\}$. The first part of the partition of the class cannot therefore exceed $\mu_1+\nu_1$. But the degree of $\phi\psi$ in the coefficient of f_1 is $\mu_1+\nu_1$. It follows that, since the

* Littlewood (4), 384.

concomitant allows the interchange of two sets of coefficients from the same ground form, the first part of the partition of the class cannot be less than $\mu_1+\nu_1$. It follows that the first part of this partition is exactly $\mu_1+\nu_1$.

Similar considerations show that the second part must be exactly $\mu_2+\nu_2$ and so on. Thus the class of $\phi\psi$ is $\{\mu_1+\nu_1,..., \mu_r+\nu_r\}$. The existence of this concomitant proves the theorem.

As an example of the use of the theorems, the following application is made to the expansion of $\{6\} \otimes \{4\}$. This expansion contains the principal parts of the products obtained from the expansions

$$(\{5\} \otimes \{4\})(\{1\} \otimes \{4\}),\quad (\{5\} \otimes \{1^4\})(\{1\} \otimes \{1^4\}),\quad \text{and}\quad (\{6\} \otimes \{2\})^2.$$

The following terms are thus obtained

$$\begin{aligned}
&\{24\}+\{22.2\}+\{21.3\}+2\{20.4\}+\{20.2^2\}+\{19.5\}+\\
&\qquad+\{19.41\}+\{19.32\}+2\{18.6\}+\{18.51\}+2\{18.42\},\\
&\{18.2^3\}+\{17.7\}+2\{17.61\}+2\{17.52\}+\{17.43\}+\\
&\qquad+\{17.421\}+2\{16.8\}+\{16.71\}+3\{16.62\}+\{16.53\},\\
&\{16.521\}+2\{16.4^2\}+\{16.42^2\}+\{15.9\}+\{15.81\}+\\
&\qquad+2\{15.72\}+\{15.71^2\}+2\{15.63\}+\{15.621\}+\{15.54\},\\
&\{15.531\}+\{15.52^2\}+\{15.4^21\}+2\{14.10\}+\{14.91\}+\\
&\qquad+2\{14.82\}+\{14.73\}+2\{14.721\}+3\{14.64\},\\
&\{14.631\}+2\{14.62^2\}+\{14.541\}+\{14.4^22\}+\{13.10.1\}+\\
&\qquad+\{13.92\}+\{13.9.1^2\}+\{13.83\}+2\{13.821\},\\
&\{13.74\}+2\{13.731\}+\{13.65\}+2\{13.641\}+\{13.632\}+\\
&\qquad+\{13.542\}+\{12^2\}+2\{12.10.2\}+\{12.93\},\\
&\{12.921\}+2\{12.84\}+\{12.831\}+2\{12.82^2\}+\{12.75\}+\\
&\qquad+2\{12.741\}+\{12.732\}+2\{12.6^2\}+\{12.651\},\\
&2\{12.642\}+\{12.4^3\}+\{11^2.1^2\}+\{11.10.3\}+\{11.10.2.1\}+\\
&\qquad+\{11.9.4\}+2\{11.9.3.1\}+\{11.85\}+2\{11.841\},\\
&\{11.832\}+\{11.76\}+2\{11.751\}+\{11.742\}+\{11.6^21\}+\\
&\qquad+\{11.652\}+\{11.643\}+\{10^2.4\}+\{10^2.2^2\},\\
&\{10.941\}+\{10.86\}+\{10.851\}+2\{10.842\}+\{10.761\}+\\
&\qquad+\{10.743\}+2\{10.6^22\}+\{10.64^2\}+\{9^23^2\},\\
&\{9861\}+\{9852\}+\{9753\}+\{96^23\}+\\
&\qquad+\{8^3\}+\{8^262\}+\{8^24^2\}+\{86^24\}+\{6^4\}.
\end{aligned}$$

The use of Littlewood's third method then leads to the following additional terms:

$$\{18.6\}+\{16.8\}+\{16.71\}+\{16.62\}+\{15.81\}+$$
$$+\{15.72\}+\{15.63\}+\{14.91\}+2\{14.82\}+\{14.73\},$$
$$\{14.6.4\}+\{13.10.1\}+\{13.9.2\}+2\{13.8.3\}+2\{13.7.4\}+$$
$$+\{13.65\}+\{12^2\}+\{12.10.2\}+\{12.93\},$$
$$2\{12.84\}+\{12.6^2\}+\{11.85\}+\{10^2.4\}+\{10.95\}+\{10.86\}.$$

This result confirms the expansion given by Foulkes,* whose method of calculation uses an unproved theorem that, for $n > m$, the forms of $\{n\} \otimes \{m\}$ are contained in the expansion of $\{m\} \otimes \{n\}$.

Another theorem conjectured but not proved by Foulkes, namely that, if $\{\lambda_1, \lambda_2,..., \lambda_n\}$ is any term in $\{m\} \otimes \{n\}$, then $\{\lambda_1+n, \lambda_2,..., \lambda_n\}$ will appear in $\{m+1\} \otimes \{n\}$, is a particular case of Theorem I of this paper.

The expansion of $\{5\} \otimes \{4\}$ which is used in the above calculation is given in Foulkes's paper.* The expansion of $\{5\} \otimes \{1^4\}$ which is also employed is as follows:

$$\{17.1\}+\{16.31\}+\{15.41\}+\{15.31^2\}+\{14.6\}+$$
$$+\{14.51\}+\{14.42\}+\{14.41^2\}+\{14.3^2\}+2\{13.61\},$$
$$\{13.52\}+2\{13.51^2\}+\{13.43\}+\{13.3^21\}+\{12.8\}+$$
$$2\{12.71\}+2\{12.62\}+2\{12.53\}+\{12.521\},$$
$$\{12.431\}+\{11.81\}+\{11.72\}+2\{11.71^2\}+2\{11.63\}+$$
$$+\{11.621\}+\{11.54\}+2\{11.531\}+\{11.3^3\},$$
$$\{10.10\}+\{10.91\}+2\{10.82\}+2\{10.73\}+\{10.721\}+$$
$$+2\{10.64\}+\{10.631\}+\{10.5^2\}+\{10.541\},$$
$$\{10.532\}+\{9^21^2\}+\{983\}+\{974\}+2\{9731\}+$$
$$+\{965\}+\{9632\}+2\{95^21\}+\{953^2\}.$$

The expansion of $\{6\} \otimes \{3\}$, which is also mentioned in the text, may be found in the tables deposited with the Royal Society Tables Committee.

* Foulkes (**1**).

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REFERENCES

1. H. O. Foulkes, *J. of London Math. Soc.* 25 (1950).
2. D. E. Littlewood, *Phil. Trans. Royal Soc.* A, 239 (1944).
3. —— ibid. 305–65.
4. —— *Proc. London Math. Soc.* (2), 49 (1947).

ON THE PRODUCT OF n LINEAR FORMS

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1. Let $L_1,..., L_n$ be n homogeneous linear forms in n variables $u_1,..., u_n$, with real coefficients and with determinant $\Delta \neq 0$; and let $c_1,..., c_n$ be any n real numbers. It was proved by Chalk (**1**) that there exist integral values of $u_1,..., u_n$ such that

$$L_1+c_1 > 0,\ L_2+c_2 > 0,...,\ L_n+c_n > 0, \tag{1}$$

$$(L_1+c_1)(L_2+c_2)...(L_n+c_n) \leqslant |\Delta|. \tag{2}$$

This result is best-possible, in the sense that it would not be true if $|\Delta|$ on the right-hand side of (2) were replaced by any smaller number. The problem of estimating the numerical value of the product on the left of (2), when the restrictions (1) are omitted, is a famous unsolved problem of Minkowski.

The main object of the present paper is to investigate what happens if, instead of restricting all the non-homogeneous forms to positive values, we restrict all but a specified one to positive values. The method used by Chalk can be adapted to cope with this problem and enables us to prove:

THEOREM 1. *There exist integers $u_1,..., u_n$ for which*

$$L_1+c_1 > 0,\ L_2+c_2 > 0,...,\ L_{n-1}+c_{n-1} > 0, \tag{3}$$

$$(L_1+c_1)...(L_{n-1}+c_{n-1})|L_n+c_n| \leqslant \tfrac{1}{2}|\Delta|. \tag{4}$$

Again the result is best-possible, as is apparent from the special example

$$L_1+c_1 = u_1,...,\ L_{n-1}+c_{n-1} = u_{n-1},\ L_n+c_n = 2u_n+1,$$

where $\Delta = 2$.

In a further paper (**2**), Chalk proved that, if $L_1,..., L_n$ satisfy a certain condition, the inequalities (1) and (2) have an infinity of solutions.

Quart. J. Math. Oxford (2), 3 (1952), 56–62.

The condition is as follows:

CONDITION A. *The forms $L_1,\ldots, L_n$ are such that not all of the n sets of equations*

$$L_2 = L_3 = \ldots = L_n = 0,\ldots,\ L_1 = L_2 = \ldots = L_{n-1} = 0 \tag{5}$$

are soluble in integers not all zero.‡

In order to prove a similar result for the inequalities (3) and (4) it is convenient to impose a further condition.

CONDITION B. *If the equations*

$$L_1 = L_2 = \ldots = L_{n-1} = 0 \tag{6}$$

have a solution in integers $u_1,\ldots, u_n$, not all zero, then the non-homogeneous linear form L_n+c_n never assumes the value zero.

I prove

THEOREM 2. *Suppose that Conditions* A *and* B *are satisfied. Then there are infinitely many sets of integers $u_1,\ldots, u_n$ for which* (3) *and* (4) *hold.*

A comment on the case $n = 2$ is made in § 6.

2. The essential lemma for the proof of both theorems is as follows.

LEMMA 1. *Let ϵ satisfy $0 < \epsilon < \frac{1}{8}$. Let $x_1,\ldots, x_n$ be homogeneous linear forms in $u_1,\ldots, u_n$, of determinant $\pm(1-\epsilon')/M$, where $0 \leqslant \epsilon' < \epsilon$ and $M > 0$. Suppose that, for all integral values of $u_1,\ldots, u_n$ for which*

$$x_1+1 > 0,\ldots,\ x_{n-1}+1 > 0, \tag{7}$$

we have
$$(x_1+1)\ldots(x_{n-1}+1)|x_n+1| \geqslant 1-\delta, \tag{8}$$

where $0 \leqslant \delta < \epsilon$. Then $M \leqslant (1-\epsilon')/(2-\delta)$.

Proof. The points $(x_1,\ldots, x_n)$ which correspond to integral values of $u_1,\ldots, u_n$ form a lattice of determinant $(1-\epsilon')/M$, and every lattice point which satisfies (7) must also satisfy (8). Since the conditions (7) are less restrictive than the corresponding conditions in Chalk's paper,§ it

‡ When Condition A is not satisfied, i.e. when each of the sets of equations (5) is soluble in integers not all zero, the forms $L_1,\ldots, L_n$ can be written in the form

$$L_i = \lambda_i(a_{i1}u_1+\ldots+a_{in}u_n),$$

where the a's are integers, and so the inequalities (3) and (4) have an infinite number of integral solutions if and only if the form L_n+c_n assumes the value zero for some integers $u_1,\ldots, u_n$.

§ Chalk (1) 218.

follows from what he proves that there is no lattice point except the origin O in the cube

$$|x_1| < 1,\ldots, |x_n| < 1. \tag{9}$$

I shall now prove that there is no lattice point except O in the parallelepiped defined by

$$|x_1| < 1,\ldots, |x_{n-1}| < 1, \tag{10}$$

$$|(1-\delta)(x_1+\ldots+x_{n-1})-x_n| < 2-\delta. \tag{11}$$

It suffices to consider points which are not in the cube (9); so, since the lattice is homogeneous, we can suppose that $x_n \leqslant -1$. Hence, by (10) and (11),

$$0 \leqslant -x_n-1 < (1-\delta)(1-x_1-\ldots-x_{n-1}).$$

By the inequality of the arithmetic and geometric means, we have

$$\begin{aligned}(x_1+1)\ldots(x_{n-1}+1)|x_n+1| &= (x_1+1)\ldots(x_{n-1}+1)(-x_n-1)\\ &< (1-\delta)(x_1+1)\ldots(x_{n-1}+1)(1-x_1-\ldots-x_{n-1})\\ &\leqslant 1-\delta.\end{aligned}$$

Since this contradicts (8), we have the result stated.

The parallelepiped defined by (10) and (11) has the volume $2^n(2-\delta)$. Applying Minkowski's fundamental theorem, we deduce that

$$2^n(2-\delta) \leqslant 2^n\frac{1-\epsilon'}{M},$$

i.e.

$$M \leqslant \frac{1-\epsilon'}{2-\delta},$$

as asserted in the enunciation.

3. Proof of Theorem 1

Without loss of generality we can suppose that $\Delta = 1$. Let M denote the lower bound of the product on the left of (4), for all integers $u_1,\ldots, u_n$ which satisfy (3). We may suppose that $M > 0$, for, if $M = 0$, there is nothing to prove.

For any $\epsilon > 0$ there exist integers $u_1^*,\ldots, u_n^*$ such that

$$L_1^*+c_1 > 0,\ldots, L_{n-1}^*+c_{n-1} > 0, \tag{12}$$

$$(L_1^*+c_1)\ldots(L_{n-1}^*+c_{n-1})|L_n^*+c_n| = \frac{M}{1-\epsilon'}, \tag{13}$$

where $0 \leqslant \epsilon' < \epsilon$. Define $x_1,\ldots, x_n$ by

$$x_i = \frac{L_i-L_i^*}{L_i^*+c_i}.$$

Then $x_1,\ldots, x_n$ are homogeneous linear forms in the integral variables $u_1-u_1^*,\ldots, u_n-u_n^*$, whose determinant is $(1-\epsilon')/M$, by (13). Since

$$x_i+1 = \frac{L_i+c_i}{L_i^*+c_i},$$

it follows from the definition of M and from (12) and (13) that, if $u_1,\ldots, u_n$ are any integers for which

$$x_1+1 > 0,\ldots,\ x_{n-1}+1 > 0,$$

then
$$(x_1+1)\ldots(x_{n-1}+1)|x_n+1| \geqslant 1-\epsilon'.$$

The hypotheses of Lemma 1 are satisfied, with $\delta = \epsilon'$. It follows that

$$M \leqslant \frac{1-\epsilon'}{2-\epsilon'}.$$

Hence either $M < \frac{1}{2}$; or $M = \frac{1}{2}$ and $\epsilon' = 0$, in which case it follows from (12) and (13) that the integers $u_1^*,\ldots, u_n^*$ satisfy (3) and (4).

4. It is convenient to make a slight change of notation before proving Theorem 2. By permuting the forms if necessary, we can ensure that

$$L_2 = L_3 = \ldots = L_n = 0 \tag{14}$$

is the set of equations in Condition A which has no non-trivial solution. But now we must consider two possibilities, according as L_1+c_1 or L_n+c_n is the form which is not restricted to positive values. Condition B tells us that, if the unrestricted form is L_n+c_n, then $L_n+c_n \neq 0$ for all sets of integers $u_1,\ldots, u_n$.

Lemma 2. *Suppose that the set of equations* (14) *has no solution in integers $u_1,\ldots, u_n$, not all zero. Then, for any $\lambda > 0$, there exist integers $u_1,\ldots, u_n$ which satisfy* (1) *and* (2) *and also satisfy*

$$(L_2+c_2)\ldots(L_n+c_n) < \lambda. \tag{15}$$

This is substantially Theorem 4 of Chalk (**2**).

The next lemma can be regarded as an imperfect form of Theorem 2, in the case when the form which is not restricted to positive values is L_n+c_n.

Lemma 3. *Suppose that the set of equations* (14) *has no non-trivial solution in integers $u_1,\ldots, u_n$. Then for any $k > \frac{1}{2}$ and any $\lambda > 0$ there exist integers $u_1,\ldots, u_n$ for which*

$$L_1+c_1 > 0,\ldots,\ L_{n-1}+c_{n-1} > 0, \tag{16}$$

$$(L_2+c_2)\ldots(L_{n-1}+c_{n-1})|L_n+c_n| < \lambda, \tag{17}$$

$$(L_1+c_1)\ldots(L_{n-1}+c_{n-1})|L_n+c_n| < k|\Delta|. \tag{18}$$

Proof. We may again suppose that $\Delta = 1$. We define $M(\lambda)$ to be the lower bound of the product on the left of (18), for all integers $u_1,\ldots, u_n$ which satisfy (16) and (17).

By Lemma 2, $M(\lambda)$ exists and satisfies $M(\lambda) \leqslant 1$. Plainly $M(\lambda)$ increases monotonically as λ decreases, and hence has a limit $M_0 \leqslant 1$ as $\lambda \to 0$. We may obviously suppose that $M_0 > 0$.

Corresponding to any $\epsilon_0 > 0$, there exist integers $u_1^*,\ldots, u_n^*$ such that (16) holds, and

$$(L_2^*+c_2)\ldots(L_{n-1}^*+c_{n-1})|L_n^*+c_n| < \epsilon_0, \tag{19}$$

and further

$$(L_1^*+c_1)\ldots(L_{n-1}^*+c_{n-1})|L_n^*+c_n| = \frac{M_0}{1-\epsilon'}, \tag{20}$$

where $0 \leqslant \epsilon' < \epsilon_0$.

We define the linear forms $x_1,\ldots, x_n$ as in the proof of Theorem 1. Their determinant is $\pm(1-\epsilon')/M_0$.

Consider any set of integers for which

$$|x_1| < 1,\ldots, |x_{n-1}| < 1, |x_n| < n+1. \tag{21}$$

By the definition of the function $M(\lambda)$ and by (20), we have

$$(x_1+1)\ldots(x_{n-1}+1)|x_n+1| = \frac{1-\epsilon'}{M_0}(L_1+c_1)\ldots(L_{n-1}+c_{n-1})|L_n+c_n|$$

$$\geqslant \frac{1-\epsilon'}{M_0} M\{2^{n-2}(n+2)(L_2^*+c_2)\ldots(L_{n-1}^*+c_{n-1})|L_n^*+c_n|\},$$

since

$$(L_2+c_2)\ldots(L_{n-1}+c_{n-1})|L_n+c_n| \leqslant 2^{n-2}(n+2)(L_2^*+c_2)\ldots(L_{n-1}^*+c_{n-1})|L_n^*+c_n|.$$

By (19), it follows that

$$(x_1+1)\ldots(x_{n-1}+1)|x_n+1| \geqslant \frac{1-\epsilon'}{M_0} M\{2^{n-2}(n+2)\epsilon_0\}.$$

Given any positive ϵ, we can choose ϵ_0 so small that $\epsilon_0 < \epsilon$ and the right-hand side of the last inequality is greater than $1-\epsilon$. The linear forms $x_1,\ldots, x_n$ now satisfy the hypotheses of Lemma 1, with M_0 in place of M, in so far as those hypotheses relate to values which satisfy (21). Since only such values are used in the proof of Lemma 1, that lemma is applicable, and, since ϵ is arbitrarily small, it follows that $M_0 \leqslant \frac{1}{2}$. Since $k > \frac{1}{2}$, it follows from the definition of $M(\lambda)$ that we can satisfy the inequalities (16), (17), (18).

LEMMA 4. *Lemma 3 remains valid if the forms L_1+c_1 and L_n+c_n are interchanged in* (16) *and* (18), *but not in* (17).

The proof is substantially the same.

5. Proof of Theorem 2

As explained at the beginning of § 4, we have to consider two cases, according as the form which is not restricted to positive values is L_1+c_1 or L_n+c_n.

Case 1. Suppose that the unrestricted form is L_n+c_n. We have to prove that there are infinitely many sets of integers $u_1,..., u_n$ which satisfy (3) and (4). By Theorem 1, there is at least one such set. Suppose there are only a finite number of such sets, and let l_i $(i = 2,..., n)$ be the least value of L_i+c_i (or $|L_n+c_n|$ if $i = n$) for all these sets. Then $l_2,..., l_{n-1}$ are positive by (3), and l_n is positive because of the hypothesis that L_n+c_n does not assume the value zero.

We choose a λ to satisfy

$$0 < \lambda < l_2 l_3...l_n/2^{n-2}(n+2).$$

Plainly, if (3) and (4) are satisfied, (17) cannot hold, since $\lambda < l_2 l_3...l_n$. Now consider the number $M(\lambda)$ defined in the proof of Lemma 3. From the preceding assertion, $M(\lambda)$ must be at least $\frac{1}{2}$. By Lemma 3,

$$M(\lambda) \leqslant M_0 \leqslant \tfrac{1}{2};$$

hence $M(\lambda) = \frac{1}{2}$.

By the definition of $M(\lambda)$, there exist for any $\epsilon > 0$, integers $u_1^\dagger,..., u_n^\dagger$ such that

$$L_1^\dagger+c_1 > 0,..., \; L_{n-1}^\dagger+c_{n-1} > 0,$$

$$(L_2^\dagger+c_2)...(L_{n-1}^\dagger+c_{n-1})|L_n^\dagger+c_n| < \lambda,$$

$$(L_1^\dagger+c_1)...(L_{n-1}^\dagger+c_{n-1})|L_n^\dagger+c_n| = \frac{1}{2(1-\epsilon')},$$

where $0 \leqslant \epsilon' < \epsilon$.

Define the linear forms $x_1,..., x_n$ in the same way as before. Consider lattice points $(x_1,..., x_n)$ for which

$$|x_1| < 1,..., \; |x_{n-1}| < 1, \; |x_n| < n+1.$$

For such points we have

$$L_1+c_1 > 0,..., \; L_{n-1}+c_{n-1} > 0,$$

and

$$\begin{aligned}(L_2+c_2)...(L_{n-1}+c_{n-1})|L_n+c_n| \\ &< 2^{n-2}(n+2)(L_2^\dagger+c_2)...(L_{n-1}^\dagger+c_{n-1})|L_n^\dagger+c_n| \\ &< l_2 l_3...l_n.\end{aligned}$$

By the definition of $l_2,..., l_n$, it follows that

$$(L_1+c_1)...(L_{n-1}+c_{n-1})|L_n+c_n| > \tfrac{1}{2};$$

whence

$$(x_1+1)...(x_{n-1}+1)|x_n+1| > 1-\epsilon'.$$

Lemma 1 becomes applicable, and, since now $M = \frac{1}{2}$, it shows that

$$\frac{1}{2} \leqslant \frac{1-\epsilon'}{2-\epsilon'}.$$

Hence $\epsilon' = 0$. This implies that the lower bound $M(\lambda)$ is attained for $u_1^\dagger,\dots, u_n^\dagger$. Since $M(\lambda) = \frac{1}{2}$, the condition (4) is satisfied, and, since $\lambda < l_2 l_3 \dots l_n$, we have a contradiction to the definition of $l_2,\dots, l_n$. This contradiction establishes the result.

Case 2. Suppose the unrestricted form to be L_1+c_1. The proof is essentially the same as that of Case 1, excepting that Lemma 4 is used in place of Lemma 3, l_n is positive since L_n+c_n is restricted to positive values, and (21) is replaced by

$$|x_1| < n+1, \quad |x_2| < 1,\dots, \quad |x_n| < 1,$$

so that it is only necessary to choose $\lambda < l_2 \dots l_n/2^{n-1}$.

6. In the case $n = 2$, Theorem 2 implies that the inequalities

$$L_1+c_1 > 0, \qquad (L_1+c_1)|L_2+c_2| \leqslant \tfrac{1}{2}|\Delta|$$

have infinitely many solutions provided that Conditions A and B are satisfied. The conditions are satisfied if

$$L_1+c_1 = \lambda_1(\phi u-v-\alpha), \qquad L_2+c_2 = \lambda_2(\theta u-v-\beta),$$

where either ϕ is irrational, or θ is irrational and $\beta \neq \theta u_0 - v_0$ for all integers u_0, v_0. It is easy to see that this last restriction on β is not necessary. However, a more precise result is known in this case since the inequalities

$$u|\theta u-v-\alpha| < 5^{-\frac{1}{2}}, \qquad u > 0,$$

where θ is irrational, have infinitely many solutions.‡

I am indebted to Professor Davenport for suggesting this problem to me, and for both his and Dr. Rogers's guidance in writing this paper.

‡ Cassels (3). Further references are given in this paper.

REFERENCES

1. J. H. H. Chalk, *Quart. J. of Math.* (Oxford), 18 (1947), 215–27.
2. —— ibid. 19 (1948), 67–80.
3. J. W. S. Cassels, *Proc. Cambridge Phil. Soc.* 44 (1948), 1–7.

MEASURE OF ASYMMETRY OF CONVEX CURVES OF CONSTANT WIDTH AND RESTRICTED RADII OF CURVATURE

By H. G. EGGLESTON (*Swansea*)

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1. Introduction

THE area of a plane point-set Δ is denoted by $|\Delta|$. The largest subset of Δ symmetric with respect to a point P is written $\Delta(P)$. The following definitions are used.†

(*a*) The *coefficient of asymmetry of* Δ *with respect to* P is

$$f(\Delta;P) = 1-|\Delta(P)|(|\Delta|)^{-1}.$$

(*b*) The *coefficient of asymmetry of* Δ is

$$g(\Delta) = \min_P f(\Delta;P).$$

It has been shown by A. S. Besicovitch in the two papers cited above that

(i) if Δ is any convex set, then $0 \leqslant g(\Delta) \leqslant \frac{1}{3}$, and these inequalities are best possible; for the case $g(\Delta) = 0$ holds when Δ is a central set and $g(\Delta) = \frac{1}{3}$ when Δ is a triangle;

(ii) if Δ is any convex set of constant width, then $0 \leqslant g(\Delta) \leqslant g(\Phi)$ where Φ is a Reuleaux triangle and $g(\Phi)$ is approximately $0{\cdot}16$.

I shall use Ω to denote the set of those points whose distances from a fixed Reuleaux triangle of side-length $2d-1$ are equal to or less than $1-d$, where $\frac{1}{2} \leqslant d < 1$. The object of this paper is to show that, if Δ is a set of constant width 1 and such that the radius of curvature at every point of δ, the frontier of Δ, exists and lies between d and $1-d$, then $g(\Delta) \leqslant g(\Omega)$.

This result was conjectured by Professor A. S. Besicovitch and I wish to express my gratitude for his encouragement and advice and for reading the manuscript of this paper. In particular the methods of proof of Lemmas 1 and 2 are due to him.

† See A. S. Besicovitch, 'Measure of asymmetry of convex curves', *J. of London Math. Soc.* 23 (1948), 237–40, and A. S. Besicovitch, 'Measure of asymmetry of convex curves, (II) Curves of constant width', ibid. 26 (1951), 81–93.

2. Notation

Greek capital letters are used for sets of points. The frontier of such a set is denoted by the corresponding small Greek letter. Other curves are denoted by other small Greek letters, and points by Roman capitals.

Let Γ_1 and Γ_2 be, respectively, the circumcircle and incircle of Δ, where by 'circumcircle' is meant the set of points on or interior to the smallest circle containing Δ, and there is an analogous meaning of 'incircle'. It is assumed that Γ_1 and Γ_2 are distinct, since otherwise Δ is a circle, $g(\Delta) = 0$, and the required inequality is true. This assumption implies that $d \neq \frac{1}{2}$. Let the common centre of γ_1 and γ_2 be O and their radii be r_1, r_2, where $r_1 > r_2$, $r_1+r_2 = 1$, and r_2 is greater than the radius of the incircle of Ω.†

The reflection of a set of points in O will be denoted by adding a dash to the symbol for the set of points concerned.

The ζ-curves

By the 'contact' of two circular arcs or two circles is meant contact such that the convexity of both arcs or circles is from the same side of the common normal at the point of contact.‡

At a point P on γ_1 is a circle touching γ_1 of radius $1-d$. Call this circle μ. There are two circles of radius d touching both μ and γ_2. Call these circles λ_1 and λ_2. Let λ_1 touch μ at P_1 and γ_2 at P_2. By 'arc PP_1' we mean the smaller of the two arcs PP_1 of μ and by 'arc P_1P_2' the smaller of the two arcs P_1P_2 of λ_1. Then the curve made up of arc PP_1 and arc P_1P_2 is a ζ-curve. Similarly another ζ-curve can be defined from λ_2.

The ζ-curves have the following properties.

(i) There are exactly two of them through each point inside the annulus between γ_1 and γ_2.

(ii) The distance of O from a variable point Q of a fixed ζ-curve varies monotonically, as the line OQ rotates in a given sense, from the value r_1 at one end-point to the value r_2 at the other. If, as OQ rotates clockwise, the distance OQ increases, we call the ζ-curve a 'clockwise ζ-curve'. In the opposite case it is called an 'anticlockwise ζ-curve'. There is one ζ-curve of each type through every point of the annulus.

(iii) Let $r_0 = \{\frac{1}{4}+(d-r_2)^2-(d-\frac{1}{2})^2\}^{\frac{1}{2}}$. Then, for a point Q of the annulus on a ζ-curve ζ_1, Q lies on the part of ζ_1 which is an arc of

† A. E. Mayer, 'Über Gleichdicke', *Z. Ver. Deutsch. Ing.* 76 (1932), 884–6 and 77 (1933), 152.

‡ This convention is also used for the contact of a circle with a convex curve.

radius d or Q lies on the part of ζ_1 which is an arc of radius $1-d$ according as the distance OQ is less than r_0 or greater than r_0. If $OQ \leqslant r_0$ and λ is a circle through Q of radius d containing Γ_2, then the non-obtuse angle between OQ and the tangent to λ at Q is greater than or equal to the non-obtuse angle between OQ and the tangent at Q to ζ_1. If $OQ \geqslant r_0$ and μ is a circle of radius $1-d$ through Q and contained in Γ_1, then the non-obtuse angle between OQ and the tangent to μ at Q is greater than or equal to the non-obtuse angle between OQ and the tangent at Q to ζ_1.

3. Proof of the relation $g(\Delta) \leqslant g(\Phi)$

On γ_1 there are at least three points of δ that do not lie on a closed semicircle of γ_1.† Let three such points be A, B, C, where ABC is in the

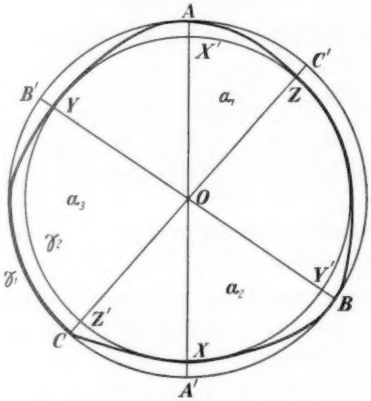

FIG. 1.

clockwise sense round γ_1 as in Fig. 1. Let AO, BO, CO meet γ_2 in the points X, Y, Z such that $AX = BY = CZ = 1$. The points X, Y, Z lie on δ.

If δ has contact with an arc ρ of radius d, then the whole of δ is on ρ or on the concave side of ρ. If δ has contact with an arc ρ of radius $1-d$, then the whole of δ is on ρ or on the convex side of ρ.‡

LEMMA 1. *If a ζ-curve meets δ in a point Q, then the points of the ζ-curve nearer than Q to O lie on δ or inside Δ, and those farther than Q from O lie on δ or outside Δ.*

† A. E. Mayer, 'Der Inhalt der Gleichdicke. Abschätzungen für ebene Gleichdicke', *Math. Annalen*, 110 (1934–5), 97–127.

‡ The first of these two statements has been proved by E. Blanc, *Ann. École Norm. Sup.* (3) 60 (1943), 224. The second is an easy deduction from the first.

I shall show that the lemma is true for ζ-curves which are anticlockwise. The proof for the clockwise ζ-curves is similar and is omitted. Let OK be a fixed line, angles be measured in the anticlockwise sense from OK, and P on δ be the point such that $OP = r$, $\angle POK = \phi$. Denote the anticlockwise ζ-curve through P by ζ_P.

At P there are two circles which touch δ: one is of radius d and contains Δ and hence also Γ_2; the other is of radius $1-d$, is contained in Δ and hence also in Γ_1. By property (iii) of the ζ-curves

$$\left(\frac{dr}{d\phi}\right)_{\zeta_P} \geqslant \left(\frac{dr}{d\phi}\right)_{\delta}.$$

But for any fixed r $dr/d\phi$ is constant on all ζ-curves ζ_P. Hence

$$\left(\frac{dr}{d\phi}\right)_{\zeta_Q} \geqslant \left(\frac{dr}{d\phi}\right)_{\delta}$$

for any equal values of r, from which it follows that, if $Q = (r_0, \phi_0)$, then

$$r_\delta(\phi) \leqslant r_{\zeta_Q}(\phi) \quad (\phi > \phi_0),$$
$$r_\delta(\phi) \geqslant r_{\zeta_Q}(\phi) \quad (\phi < \phi_0),$$

where $r_\delta(\phi)$ is the distance from O of the point P on δ such that $\angle POK = \phi$, and $r_{\zeta_Q}(\phi)$ is the distance from O of the point R on ζ_Q such that $\angle ROK = \phi$.

Since r is an increasing function of ϕ on ζ_Q, these inequalities imply the statement of the lemma.

COROLLARY. *Suppose that δ and δ' meet in a point P, and that ζ_1, ζ_2 are the two ζ-curves through P. Then the curvilinear triangle bounded by arcs of ζ_1, ζ_2, γ_1 is exterior to both Δ and Δ', whilst the curvilinear triangle bounded by arcs of ζ_1, ζ_2, γ_2 is interior to both Δ and Δ'.*

Now denote the area of the part of Δ in the sector AOC' by a and the area of the part of Δ in the sector $A'OC$ by b. Consider a variable point P on that arc AZ of δ which is contained in the sector AOC'. Let the anticlockwise ζ-curve through P have end-points E on γ_2 and H on γ_1. Then the arc AH of γ_1, the ζ-curve HPE, the arc EZ of γ_2, the linear segment ZO, and the linear segment OA together bound a convex set whose area $h(P)$ has the following properties:

(i) $h(P)$ varies continuously with P;

(ii) $h(Z) \geqslant a$, $h(A) \leqslant a$.

The property (ii) is a consequence of Lemma 1. It follows that there exists a point P on AZ such that $h(P) = a$. For this P, the curve consisting successively of the arc AH of γ_1, the ζ-curve HPE, and the arc EZ of γ_2 is called α_1.

Similarly in sector $A'OC$ construct a curve β_1 using a clockwise ζ-curve so that β_1, XO, OC enclose an area b. Repeat the process in sectors BOA', COB' to obtain curves α_2, α_3 and in sectors $B'OA$, $C'OB$ to obtain curves β_2, β_3 respectively.

The closed convex curve made up successively of $\alpha_1, \beta_2, \alpha_3, \beta_1, \alpha_2, \beta_3$ is called δ_1, and the set which it bounds is called Δ_1; δ_1 may be denoted by

$$AD_1D_2ZD_3D_4BD_5D_6XD_7D_8CD_9D_{10}YD_{11}D_{12}A,$$

where

$D_{12}AD_1$, D_4BD_5, D_8CD_9 are arcs or points of γ_1,

D_2ZD_3, D_6XD_7, $D_{10}YD_{11}$ are arcs or points of γ_2,

D_1D_2, D_5D_6, D_9D_{10} are anticlockwise ζ-curves,

D_3D_4, D_7D_8, $D_{11}D_{12}$ are clockwise ζ-curves.

LEMMA 2. $f(\Delta; 0) \leqslant f(\Delta_1; 0)$.

By the definition of Δ_1, $|\Delta_1| = |\Delta|$ and thus, in order to prove the lemma, it is sufficient to show that $|\Delta(0)| \geqslant |\Delta_1(0)|$.

Consider the intersection of the parts of Δ and Δ_1 in the sector AOC' with the reflections in O of the parts of Δ and Δ_1 respectively in $A'OC$. It is impossible for α_1 and β_1' to intersect in an arc or point of γ_1, for, if they did, we should have

$$a+b > \tfrac{1}{2}(r_1^2+r_2^2)(\angle AOC'),$$

whereas in fact

$$a+b = \tfrac{1}{2}\int_0^{\angle AOC'} \{(r(\theta))^2+(r(\theta+\pi))^2\}\,d\theta$$

$$\leqslant \tfrac{1}{2}\int_0^{\angle AOC'} \{(r(\theta))^2+(1-r(\theta))^2\}\,d\theta$$

$$\leqslant \tfrac{1}{2}(r_1^2+r_2^2)(\angle AOC'),$$

where $r(\theta)$ is the distance from O of a point P on δ such that $\angle POC' = \theta$.

Thus there are two cases to consider.

(i) α_1 intersects β_1' in an arc or point of γ_2. In this case the common part of Δ_1 and Δ_1' inside the sector AOC' lies inside Γ_2 and is therefore not more than the common part of Δ and Δ' in this sector.

(ii) α_1 and β_1' meet in a point interior to the annulus. Let this point be K and L be any point of intersection of δ and δ' that lies in the sector AOC'. It is clear that there is at least one such point L.

For every point P in the annulus between γ_1 and γ_2, the two arcs of ζ-curves through P and the smaller of the two arcs of γ_1 which join two

of the end-points of the ζ-curves, bound a set of points whose area is written $k_1(P)$. Similarly the two arcs of ζ-curves through P and the smaller of the two arcs of γ_2 which join two of their end-points bound an area $k_2(P)$. Both $k_1(P)$ and $k_2(P)$ are functions of OP only, $k_1(P)$ a decreasing function and $k_2(P)$ an increasing function.

By the corollary to Lemma 1, the area common to both Δ and Δ' in the sector AOC' is greater than or equal to both $k_2(L)+\frac{1}{2}r_1^2(\angle AOC')$ and $a+b+k_1(L)-\frac{1}{2}r_1^2(\angle AOC')$. Similarly the area common to both Δ_1 and Δ_1' in AOC' is equal to both $k_2(K)+\frac{1}{2}r_1^2(\angle AOC')$ and

$$a+b+k_1(K)-\tfrac{1}{2}r_1^2(\angle AOC').$$

Either $k_2(K) \leqslant k_2(L)$ or $k_1(K) \leqslant k_1(L)$. Thus in all cases the area of $\Delta.\Delta'$ in AOC' is greater than or equal to the area of $\Delta_1.\Delta_1'$ in the sector AOC'. Similar relations hold for all the other sectors. Thus

$$|\Delta(0)| \geqslant |\Delta_1(0)|.$$

This proves the lemma.

Denote the sum of the lengths of the arcs of γ_1, $D_{12}AD_1$, D_4BD_5, D_8CD_9 by l and the sum of the lengths of the arcs of γ_2, D_2ZD_3, D_6XD_7, $D_{10}YD_{11}$ by m. Construct a set Δ_2 whose boundary δ_2 is made up of a similar succession of arcs to that which forms δ_1. Here

δ_2 is $$E_1E_2E_3E_4E_5E_6E_7E_8E_9E_{10}E_{11}E_{12}E_1,$$
where

$E_{12}E_1$, E_4E_5, E_8E_9 are arcs of γ_1 of length $\frac{1}{3}l$,

E_2E_3, E_6E_7, $E_{10}E_{11}$ are arcs of γ_2 of length $\frac{1}{3}m$,

E_1E_2, E_5E_6, E_9E_{10} are anticlockwise ζ-curves,

E_3E_4, E_7E_8, $E_{11}E_{12}$ are clockwise ζ-curves.

Δ_2 is symmetric with respect to each of three axes through O making angles of 60° with one another.

We have

LEMMA 3. (i) $|\Delta_2| = |\Delta_1|$. (ii) $|\Delta_2(0)| \leqslant |\Delta_1(0)|$.

(i) Suppose PQ is a ζ-curve with end-points P, Q. Together with linear segments OP, OQ it forms the boundary of a set whose area is (say) c. This area is the same for all positions of PQ. Hence

$$|\Delta_1| = \tfrac{1}{2}r_1l+\tfrac{1}{2}r_2m+6c = |\Delta_2|.$$

(ii) Consider a variable point X on a fixed ζ-curve PQ where P lies on γ_2 and Q on γ_1 and let $\angle XOP$ be χ measured positively, so that when X is at Q, χ has a positive value. Denote the area contained in the contour XO, OP and arc PX of the ζ-curve by $j(\chi)$. For $\chi < 0$

define $j(\chi)$ to be $\frac{1}{2}r_2^2\chi$, for $\chi > \angle QOP$ define $j(\chi)$ to be $\frac{1}{2}r_1^2\chi$. Then

$$\frac{dj(\chi)}{d\chi} = \begin{cases} \frac{1}{2}r_2^2 & (\chi < 0), \\ \frac{1}{2}OX^2 & (0 \leqslant \chi \leqslant \angle QOP), \\ \frac{1}{2}r_1^2 & (\chi > \angle QOP). \end{cases}$$

Since OX is an increasing function of χ, it follows that $dj(\chi)/d\chi$ is a non-decreasing function of χ, so that $j(\chi)$ is a convex function of χ.

Now the intersection of arcs AD_1D_2Z of δ_1 and $X'D_7'D_8'C'$ of δ_1' may occur in one of two different ways (see the proof of Lemma 2);

(i) in a point of ζ-curve D_1D_2, (ii) in an arc or point of γ_2.

In case (i) the point is called X_1; in case (ii) the mid-point of arc D_2D_7' of γ_2 is called X_1. Similarly define X_2 on ZD_3D_4B and $Y'D_{10}'D_9'C'$; X_3 on BD_5D_6X and $Y'D_{11}'D_{12}'A'$. Let $\angle X_1OD_2 = \chi_1$, $\angle X_2OD_3 = \chi_2$, $\angle X_3OD_6 = \chi_3$.

The part of $\Delta_1(0)$ in the sector D_2OD_7' is symmetric about OX_1 and similarly for OX_2, OX_3, while the set $\Delta_1(0)$ is symmetric about O. Thus

$$|\Delta_1(0)| = r_2m + 4\{j(\chi_1)+j(\chi_2)+j(\chi_3)\}.$$

Since the function $j(\chi)$ is convex, it follows that

$$|\Delta_1(0)| \geqslant r_2m + 12j\{\tfrac{1}{3}(\chi_1+\chi_2+\chi_3)\}.$$

It is easy to see that this last expression is precisely $|\Delta_2(0)|$.

This completes the proof of Lemma 3.

It follows from Lemma 3 that $f(\Delta_2; 0) \geqslant f(\Delta_1; 0)$. We are now in a position to construct a set Δ_3 which can be compared both with Δ_2 and with Ω. Let Ω be considered to lie so that it is symmetric about the same three axes through O as is Δ_2, and so that the direction in which a half-line through O meets ω in points whose distance from O is the largest possible, has this same property with respect to δ_2.

Let two of the axes of symmetry through O be OG_1 and OH_1 so that $\angle G_1OH_1$ is 60°. Let δ_2 in this sector be $G_2E_1E_2H_2$, where G_2 lies on OG_1 and H_2 lies on OH_1. Let ω in this sector be G_3H_3. See Fig. 2. On the ζ-curve E_1E_2 let F be the point at which the radius of curvature changes from d to $1-d$. Let J be a similar point on G_3H_3.

Now, denote the circumcircle and incircle of Ω by σ_1 and σ_2 respectively, and construct the curve $G_3K_2K_3H_4$ where G_3 is the point of σ_1 on the half-line OG_1, G_3K_2 is an arc of σ_1 which may reduce to a point, K_2K_3 is a circular arc of radius $1-d$ touching σ_1 internally at K_2, K_3H_4 is a circular arc of radius d touching K_2K_3 at K_3 and intersecting OH orthogonally at H_4, and where also the area enclosed by $G_3K_2K_3H_4OG_3$ is equal to the area enclosed by $G_2E_1FE_2H_2OG_2$.

Let Δ_3 be the set bounded by $G_3 K_2 K_3 H_4$ and by the five other curves that can be obtained from this one by symmetry in the three axes of symmetry through O.

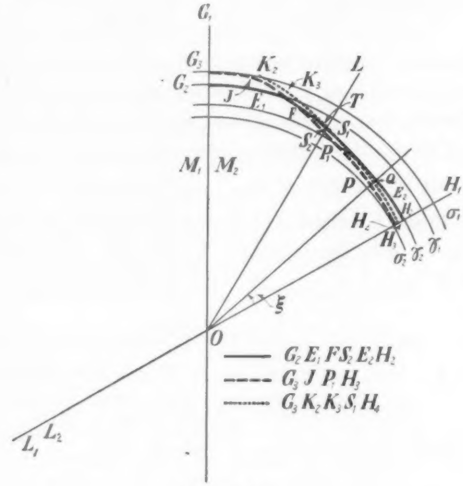

Fig. 2.

Lemma 4. (i) $|\Delta_3| = |\Delta_2| \geqslant |\Omega|$. (ii) $f(\Delta_3; 0) \geqslant f(\Delta_2; 0)$.

(i) An immediate consequence of the definition of Δ_3 is that $|\Delta_3| = |\Delta_2|$. Now $|\Delta_2| = |\Delta_1| = |\Delta|$, and it is known that of all curves of width 1 the curve ω contains the least area.† This proves (i).

(ii) To prove $f(\Delta_3; 0) \geqslant f(\Delta_2; 0)$: since $|\Delta_3| = |\Delta_2|$, it is sufficient to show that $|\Delta_3 . \Delta_3'| \leqslant |\Delta_2 . \Delta_2'|$.

The curves $G_3 K_2 K_3 H_4$ and $G_2 E_1 F E_2 H_2$ cut in exactly one point. It is clear that these two curves have at least one point of intersection. No point of intersection can lie on $G_3 K_2$. $K_2 H_4$ is a ζ-curve defined in the annulus formed by the circle σ_1 and the circle θ whose centre is O and whose radius is OH_4.

Although $G_2 E_1 E_2 H_2$ is not part of the boundary of a set of constant width, it is made up of circular arcs of radii r_1, r_2, d, and $1-d$, and the method of proof of Lemma 1 applies. There is one point or one arc of $G_3 K_2 K_3 H_4$ on $G_2 E_1 F E_2 H_2$. The second possibility can only arise if the arcs FE_2 and $K_3 H_4$ have some part in common. This can be so only if H_4 is at H_2, and this implies that γ_2 is σ_2 and γ_1 is σ_1. This case has

† T. Bonnesen and W. Fenchel, 'Theorie der konvexen Körper', *Ergebnisse der Math.* (Berlin, 1934), 132.

been ruled out as it was assumed initially that γ_2 has radius greater than that of σ_2.

Let the point of intersection of $G_3 K_2 K_3 H_4$ and $G_2 E_1 F E_2 H_2$ be T. Let OL be the bisector of angle $G_1 O H_1$, meeting $G_3 H_4$ in S_1 and $G_2 H_2$ in S_2. Now a variable point X on $G_3 H_4$ is such that OX is monotonic non-decreasing as X moves from H_4 to G_3. Hence

$$|\Delta_3 . \Delta_3'| = 12 \times \text{area bounded by } S_1 O H_4 S_1.$$

Similarly $$|\Delta_2 . \Delta_2'| = 12 \times \text{area bounded by } S_2 O H_2 S_2.$$

If T lies in or on the boundary of the sector LOG_1,

$$\text{area bounded by } S_1 O H_4 S_1 \leqslant \text{area bounded by } S_2 O H_2 S_2.$$

If T lies in the sector LOH_1,

$$\text{area bounded by } G_3 O S_1 K_2 G_3 \geqslant \text{area bounded by } G_2 O S_2 G_2,$$

and this in turn implies that

$$\text{area bounded by } S_1 O H_4 S_1 \leqslant \text{area bounded by } S_2 O H_2 S_2.$$

Thus in any case $|\Delta_3 . \Delta_3'| \leqslant |\Delta_2 . \Delta_2'|$ and this proves the lemma.

LEMMA 5. $f(\Delta_3; 0) \leqslant f(\Omega; 0)$.

Because of Lemma 4 (i) the curve $G_3 J H_3$ lies inside the area $O G_3 K_2 K_3 H_4 O$. Let a variable half-line through O meet $G_3 J H_3$ in P and $G_3 K_2 K_3 H_4$ in Q. I show first that OQ/OP decreases as P moves along $H_3 J G_3$ from H_3 to G_3.

Let $OP = x$, and angle POH_1 be ξ. Then, if $\epsilon(P)$ is the angle subtended at P by O and the centre of the circular arc on which P lies,

$$\frac{dx}{d\xi} = x \tan \epsilon(P).$$

Thus, to prove the required result we only have to show that $\epsilon(Q) \leqslant \epsilon(P)$.

Divide the range of ξ, $0 \leqslant \xi \leqslant \frac{1}{3}\pi$, into three or four parts, as follows. If $\angle K_2 O H_1 \geqslant \angle J O H_1$, then take the division

(i) $0 \leqslant \xi \leqslant \angle K_3 O H_1$, (ii) $\angle K_3 O H_1 \leqslant \xi \leqslant \angle J O H_1$,

(iii) $\angle J O H_1 \leqslant \xi \leqslant \angle K_2 O H_1$, (iv) $\angle K_2 O H_1 \leqslant \xi \leqslant \angle G_1 O H_1$.

If $\angle K_2 O H_1 < \angle J O H_1$, replace (ii), (iii), (iv) by

(ii′) $\angle K_3 O H_1 \leqslant \xi \leqslant \angle K_2 O H_1$ and (iii′) $\angle K_2 O H_1 \leqslant \xi \leqslant \angle G_1 O H_1$.

We consider the first case only, since the second is similar but simpler.

(i) Let L_1, L_2 be the centres of the circles of which the circular arcs $J H_3$ and $K_3 H_4$ are respectively parts. L_1 and L_2 lie on the same side of OPQ or on OPQ. $OL_1 \geqslant OL_2$, $OP \leqslant OQ$, and so either $L_1 P$ meets

$L_2 Q$ in the obtuse sector $L_1 OQ$ or $L_1 P$ coincides with $L_2 Q$. In either case

$$\epsilon(P) = \angle L_1 PO \geqslant \angle L_2 QO = \epsilon(Q).$$

(ii) Let M_2 be the centre of the arc $K_2 K_3$. QM_2 meets OL_1 in L_2 when Q is at K_3, and QM_2 meets OL_1 in O when Q is at K_2. Thus, for Q on $K_2 K_3$, QM_2 meets OL_1 in a point of the line segment OL_1. Also M_2 and L_1 lie on the same side of OPQ and $OQ \geqslant OP$. Thus, for

$$\angle K_3 OH_1 \leqslant \xi \leqslant \angle JOH_1,$$

$$\epsilon(P) = \angle L_1 PO \geqslant \angle M_2 QO = \epsilon(Q).$$

(iii) Let M_1 be the centre of arc $G_3 J$. Then M_2 is inside the triangle $OM_1 P$ and Q is a point outside this triangle on OP. Hence $\epsilon(P) \geqslant \epsilon(Q)$.

(iv) On $G_3 K_2$, OQ is constant, OP increases as ξ increases. Hence OQ/OP is decreasing.

This proves the required result. OL meets $G_3 K_2 K_3 H_4$ in S_1 and let it meet $G_3 JH_3$ in P_1. Let $OP_1/OS_1 = e$, and denote by Δ_4 the set obtained from Δ_3 by a similitude with O as centre of similarity and linear dimensions reduced in the ratio $e:1$. Then Δ_4 is bounded by a curve that lies outside Ω in the sector LOH_1 and inside Ω in the sector LOG_1. Also the distance of a point of this curve from O is non-decreasing as the point moves on the curve inside the sector from a point of OH_1 to a point of OG_1.

Thus $$f(\Delta_3; 0) = f(\Delta_4; 0) \leqslant f(\Omega; 0).$$

This completes the proof of Lemma 5.

From Lemmas 2, 3, 4, 5, $f(\Delta; 0) \leqslant f(\Omega; 0)$. And thus $g(\Delta) \leqslant f(\Omega; 0)$. The theorem will be proved if it can be shown that $f(\Omega; 0) = g(\Omega)$. This, however, follows trivially from the two facts:

(i) for each convex set, say Π, there is a point P for which

$$f(\Pi; P) = g(\Pi);$$

(ii) if Π is symmetrical about a line, then the point P of (i) may be taken to lie on that line.

This concludes the proof of the statement given in the introduction.

GENERAL TRANSFORMATIONS OF BILATERAL SERIES

By L. J. SLATER (*London*)

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1. SEARS† has given the general theorem for basic hypergeometric series

$$\Pi\begin{pmatrix} xA, q/xA, b_1, b_2,..., b_M, q/a_{M+2},..., q/a_{M+N+1}; \\ a_1, a_2,..., a_{M+1}, q/b_{M+1},..., q/b_{M+N}; \end{pmatrix}\times$$

$$\times {}_{M+N+1}\Phi_{M+N}\begin{bmatrix} a_1, a_2,..., a_{M+N+1}; & x \\ b_1, b_2,..., b_{M+N}; & \end{bmatrix}$$

$$= \Pi\begin{pmatrix} a_1 xA, q/a_1 xA, b_1/a_1,..., b_M/a_1, qa_1/a_{M+2},..., qa_1/a_{M+N+1}; \\ a_1, a_2/a_1,..., a_{M+1}/a_1, qa_1/b_{M+1},..., qa_1/b_{M+N}; \end{pmatrix}\times$$

$$\times {}_{M+N+1}\Phi_{M+N}\begin{bmatrix} a_1, qa_1/b_1, qa_1/b_2,..., qa_1/b_{M+N}; & \dfrac{qb_1 b_2...b_M}{xAa_1 a_2...a_{M+1}} \\ qa_1/a_2, qa_1/a_3,..., qa_1/a_{M+N+1}; & \end{bmatrix}+$$

$$+\text{idem}(a_1; a_2, a_3,..., a_{M+1})-$$

$$-\Pi\begin{pmatrix} b_{M+1} xA/q, q^2/b_{M+1} xA, b_{M+1}/a_{M+2},..., b_{M+1}/a_{M+N+1}, \\ qb_1/b_{M+1},..., qb_M/b_{M+1}; \\ qa_1/b_{M+1},..., qa_{M+1}/b_{M+1}, b_{M+1}/q, b_{M+1}/b_{M+2},..., b_{M+1}/b_{M+N}; \end{pmatrix}\times$$

$$\times {}_{M+N+1}\Phi_{M+N}\begin{bmatrix} qa_1/b_{M+1}, qa_2/b_{M+1},..., qa_{M+N+1}/b_{M+1}; & x \\ q^2/b_{M+1}, qb_1/b_{M+1},..., qb_{M+N}/b_{M+1}; & \end{bmatrix}-$$

$$-\text{idem}(b_{M+1}; b_{M+2}, b_{M+3},..., b_{M+N}) \quad (1)$$

provided that $|x| < 1$, $|q| < 1$, where

$$A \equiv \frac{a_{M+2}\, a_{M+3} \cdots a_{M+N+1}}{b_{M+1}\, b_{M+2} \cdots b_{M+N}}, \qquad \Pi\begin{pmatrix} a; \\ b; \end{pmatrix} \equiv \prod_{n=0}^{\infty} \frac{(1-aq^n)}{(1-bq^n)},$$

$${}_{M+1}\Phi_M\begin{bmatrix} a_1, a_2,..., a_{M+1}; & x \\ b_1, b_2,..., b_M; & \end{bmatrix} \equiv \sum_{n=0}^{\infty} \frac{(a_1)_n (a_2)_n \cdots (a_{M+1})_n\, x^n}{(q)_n (b_1)_n \cdots (b_M)_n},$$

$$(a)_n \equiv (1-a)(1-aq)(1-aq^2)...(1-aq^{n-1}), \quad (a)_0 \equiv 1,$$

and 'idem$(a; b)$' means that the preceding expression is to be repeated with b written in place of a. This theorem expresses the general series ${}_{M+N+1}\Phi_{M+N}(x)$ in terms of $M+N+1$ other series of the same type.

† Sears (4) (4.3).

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For the bilateral series we have the usual notation†

$${}_M\Psi_M\begin{bmatrix}a_1, a_2,..., a_M; \\ b_1, b_2,..., b_M;\end{bmatrix} x \equiv \sum_{n=-\infty}^{\infty} \frac{(a_1)_n(a_2)_n...(a_M)_n}{(b_1)_n(b_2)_n...(b_M)_n} x^n$$

where, for convergence, $|x| < 1$, $|b_1 b_2...b_M| < |a_1 a_2...a_M x|$, and $(a)_{-n} \equiv 1/[(1-a/q)(1-a/q^2)...(1-a/q^n)]$.

Since $$(a)_{-n} = \frac{(-1)^n q^{\frac{1}{2}n(n+1)}}{a^n(q/a)_n},$$

the order of summation of any of the Ψ series can be reversed, i.e.

$${}_M\Psi_M\begin{bmatrix}a_1, a_2,..., a_M; \\ b_1, b_2,..., b_M;\end{bmatrix} x = {}_M\Psi_M\begin{bmatrix}q/b_1, q/b_2,..., q/b_M; & \dfrac{b_1 b_2...b_M}{a_1 a_2...a_M x}\\ q/a_1, q/a_2,..., q/a_M; \end{bmatrix}.$$

Suppose that, in (1), we put $b_{M+1} = a_1, b_{M+2} = a_2,..., b_{M+N} = a_N$, $a_{N+1} = q$, and let $N = M$. Then all the series ${}_{M+N+1}\Phi_{M+N}(x)$ reduce to series of the type ${}_{M+1}\Phi_M(x)$, and we can write (1) in the form

$$P\Phi = P(a_1)\Phi(a_1)+\text{idem}(a_1; a_2, a_3,..., a_M)+P(q)\Phi(q)-$$
$$-P(b_{M+1})\Phi(b_{M+1})-\text{idem}(b_{M+1}; b_{M+2}, b_{M+3},..., b_{2M}), \quad (2)$$

where $P(a_1)$, etc., are the products preceding the series $\Phi(a_1)$, etc. If we write $-n-1$ for n in the series $\Phi(q)$, since

$$\sum_{n=0}^{\infty} f(n) = \sum_{n=-\infty}^{-1} f(-n-1),$$

we can reverse $\Phi(q)$. Denoting the reversed series by $\Phi(q)'$, we find that $\Phi(q) = K\Phi(q)'$, where K is some constant independent of n. Also, $P(q) = -P/K$. Thus,

$$P\Phi-P(q)\Phi(q) = P[\Phi+\Phi(q)'], \quad (3)$$

and the series Φ and $\Phi(q)'$ can be combined together to form a bilateral series. Similarly, we find that each of the series $\Phi(b_{M+i})$ combines with one of the series $\Phi(a_i)$ for $i = 1, 2,..., M$, to form a bilateral series $\Psi(a_i)$, and, writing c_1 for a_{M+2}, c_2 for a_{M+3},..., c_M for a_{2M+1}, we have the general theorem:

$$\Pi\begin{pmatrix}xA, q/xA, b_1, b_2,..., b_M, q/c_1,..., q/c_M; \\ a_1, a_2,..., a_M, q/a_1,..., q/a_M;\end{pmatrix} \times {}_M\Psi_M\begin{bmatrix}c_1, c_2,..., c_M; \\ b_1, b_2,..., b_M;\end{bmatrix} x$$
$$= \frac{q}{a_1}\Pi\begin{pmatrix}a_1 xA/q, q^2/a_1 xA, a_1/c_1,..., a_1/c_M, qb_1/a_1,..., qb_M/a_1; \\ a_1, q/a_1, a_1/a_2,..., a_1/a_M, qa_2/a_1,..., qa_M/a_1;\end{pmatrix}\times$$
$$\times {}_M\Psi_M\begin{bmatrix}qc_1/a_1, qc_2/a_1,..., qc_M/a_1; \\ qb_1/a_1, qb_2/a_1,..., qb_M/a_1;\end{bmatrix} x +$$
$$+\text{idem}(a_1; a_2, a_3,..., a_M), \quad (4)$$

† Bailey (1) § 4.

where now $A \equiv c_1 c_2 \dots c_M / a_1 a_2 \dots a_M$, $|x| < 1$, $|q| < 1$. This expresses the general ${}_M\Psi_M(x)$ series in terms of M other series of the same type.

If we put $b_M = q$, the Ψ series on the left of (4) becomes a ${}_M\Phi_{M-1}$ series. If we put $a_1 = b_1$, $a_2 = b_2, \dots, a_Q = b_Q$, the first Q ${}_M\Psi_M$ series become ${}_M\Phi_{M-1}(x)$ series. Similarly, if we reverse the last $M-R$ bilateral series and put $qc_{R+1} = a_{R+1}, \dots, qc_M = a_M$, these series become ${}_M\Phi_{M-1}[b_1 \dots b_M / c_1 \dots c_M x]$ series. Carrying out all these three processes, when $R = Q$, (4) reduces to a restatement of (1).

If $M = 2$, $d = q$, and $x = c/ab$, in (4), the first series is summable by Gauss's analogue,† and we have

$$\Pi\begin{pmatrix} c/ef, qef/c, q, q/a, q/b, c/a, c/b; \\ e, f, q/e, q/f, c/ab; \end{pmatrix}$$
$$= \frac{q}{e}\Pi\begin{pmatrix} c/qf, q^2f/c, e/a, e/b, qc/e, q^2/e; \\ e, q/e, e/f, qf/e; \end{pmatrix} {}_2\Psi_2\begin{bmatrix} e/c, e/q; \\ e/a, e/b; \end{bmatrix} q + $$
$$+\mathrm{idem}(e; f). \quad (5)$$

This is a relation between two ${}_2\Psi_2$ series.

If $M = 3$, and $a_1 = d$, $a_2 = e$, $a_3 = f$, in (4), similarly, we obtain a relation which expresses a general ${}_3\Psi_3(x)$ series in terms of three ${}_3\Phi_2(x)$ series.

2. Sears has also given the following theorem‡ for well-poised basic hypergeometric series,

$$\Pi\begin{pmatrix} qa_1/a_{M+2}, \dots, qa_1/a_{2M}, q/a_{M+2}, \dots, q/a_{2M}, \surd a_1, -\surd a_1, q/\surd a_1, -q/\surd a_1; \\ a_1, \dots, a_{M+1}, a_2/a_1, \dots, a_{M+1}/a_1; \end{pmatrix} \times$$
$$\times {}_{2M}\Phi_{2M-1}\begin{bmatrix} a_1, a_2, \dots, a_{2M}; & \dfrac{-q^M a_1^{M-1}}{a_2 a_3 \dots a_{2M}} \\ qa_1/a_2, \dots, qa_1/a_{2M}; & \end{bmatrix}$$
$$= a_2 \Pi\begin{pmatrix} qa_2/a_{M+2}, \dots, qa_2/a_{2M}, qa_1/a_2 a_{M+2}, \dots, qa_1/a_2 a_{2M}, \\ \surd a_1/a_2, -\surd a_1/a_2, qa_2/\surd a_1, -qa_2/\surd a; \\ a_1/a_2, a_2, a_3/a_2, \dots, a_{M+1}/a_2, a_2^2/a_1, a_2 a_3/a_1, \dots, a_2 a_{M+1}/a_1; \end{pmatrix} \times$$
$$\times {}_{2M}\Phi_{2M-1}\begin{bmatrix} a_2, a_2^2/a_1, a_2 a_3/a_1, \dots, a_2 a_{2M}/a_1; & \dfrac{-q^M a_1^{M-1}}{a_2 a_3 \dots a_{2M}} \\ qa_2/a_1, qa_2/a_3, \dots, qa_2/a_{2M}; & \end{bmatrix} +$$
$$+\mathrm{idem}(a_2; a_3, a_4, \dots, a_{M+1}). \quad (6)$$

This expresses a well-poised ${}_{2M}\Phi_{2M-1}$ series in terms of M other well-poised ${}_{2M}\Phi_{2M-1}$ series. Let us suppose that M is odd, i.e. $M = 2N+1$, and let $a_1 = a$, $a_2 = q$, $a_3 = qa_1/a_4$, $a_5 = qa_1/a_6, \dots, a_M = qa_1/a_{M+1}$. Write α_1 for a_4, α_2 for $a_6, \dots, \alpha_N$ for a_{M+1}, b_1 for a_{M+2}, b_2 for $a_{M+3}, \dots, b_{2N}$

† Bailey (2) § 8.4, (3). ‡ Sears (4) (7.2).

for a_{2M}. There are $4N+2$ parameters in all. We can again combine the series together in pairs, and we have, after some reduction,

$$\Pi\begin{pmatrix} aq/b_1,\ldots, aq/b_{2N}, q/b_1,\ldots, q/b_{2N}, \sqrt{a}, -\sqrt{a}, q/\sqrt{a}, -q/\sqrt{a}; \\ a, a_1,\ldots, a_N, aq/a_1,\ldots, aq/a_N, q/a, q/a_1,\ldots, q/a_N, a_1/a,\ldots, a_N/a; \end{pmatrix}\times$$
$$\times\, {}_{2N}\Psi_{2N}\left[\begin{matrix} b_1, b_2,\ldots, b_{2N}; \\ qa/b_1, qa/b_2,\ldots, qa/b_{2N}; \end{matrix}\ \frac{-a^N q^N}{b_1 b_2\ldots b_{2N}}\right]$$
$$= a_1\,\Pi\begin{bmatrix} a_1 q/b_1,\ldots, a_1 q/b_{2N}, aq/a_1 b_1,\ldots, aq/a_1 b_{2N}, \sqrt{a}/a_1, -\sqrt{a}/a_1, \\ qa_1/\sqrt{a}, -qa_1/\sqrt{a}; \\ a/a_1, q/a_1, a_1, a_2/a_1,\ldots, a_N/a_1, a_1^2/a, qa/a_1^2, qa_1/a, a_1 a_2/a,\ldots, \\ a_1 a_N/a, q/a_2, qa_1/a_3,\ldots, qa_1/a_N, qa/a_1 a_2,\ldots, qa/a_1 a_N; \end{bmatrix}\times$$
$$\times\, {}_{2N}\Psi_{2N}\left[\begin{matrix} a_1 b_1/a, a_1 b_2/a,\ldots, a_1 b_{2N}/a; \\ qa_1/b_1, qa_1/b_2,\ldots, qa_1/b_{2N}; \end{matrix}\ \frac{-a^N q^N}{b_1 b_2\ldots b_{2N}}\right]+$$
$$+\mathrm{idem}(a_1; a_2, a_3,\ldots, a_N). \tag{7}$$

This expresses a well-poised ${}_{2N}\Psi_{2N}$ series in terms of N other well-poised ${}_{2N}\Psi_{2N}$ series. If we let $a_1 = b_{N+1}$, $a_2 = b_{N+2},\ldots, a_Q = b_{N+Q}$, we get a well-poised ${}_{2N}\Psi_{2N}$ series expressed in terms of Q ${}_{2N}\Phi_{2N-1}$ series and $N-Q$ ${}_{2N}\Psi_{2N}$ series. If, further, $Q = N$, and $b_N = a$, (7) reduces to (6) again. If we suppose M odd, in (6) we only obtain the same result as if we had supposed M even, and put $a_N = b_{2N}$. When $a_1 = b_1 = q\sqrt{a}$, and $a_2 = b_2 = -q\sqrt{a}$, two of the series on the right of (7) vanish and the theorem then expresses a well-poised ${}_{2N}\Psi_{2N}$ in terms of $N-2$ well-poised ${}_{2N}\Psi_{2N}$ series, all with the special forms of the first and second parameters. The special cases $N = 3$, $N = 4$, of this result have already been given by M. Jackson.†

Similarly, from Sears's other results‡ we obtain

$$\Pi\begin{pmatrix} aq/b_1,\ldots, aq/b_{2N}, q/b_1,\ldots, q/b_{2N}; \\ q, a_1,\ldots, a_{N+1}, aq/a_1,\ldots, aq/a_{N+1}, q/a_1,\ldots, q/a_{N+1}, a_1/a,\ldots, a_{N+1}/a; \end{pmatrix}\times$$
$$\times\, {}_{2N}\Psi_{2N}\left[\begin{matrix} b_1,\ldots, b_{2N}; \\ aq/b_1,\ldots, aq/b_{2N}; \end{matrix}\ \frac{q^N a^N}{b_1 b_2\ldots b_{2N}}\right]$$
$$= \Pi\begin{pmatrix} a_1 q/b_1,\ldots, a_1 q/b_{2N}, aq/a_1 b_1,\ldots, aq/a_1 b_{2N}; \\ a_1, a_2/a_1,\ldots, a_{N+1}/a_1, q/a_1, qa/a_1, a_1/a, qa/a_1 a_2,\ldots, qa/a_1 a_{N+1}, \\ a_1 a_2/a,\ldots, a_1 a_{N+1}/a, a_1 q/a_2,\ldots, a_1 q/a_{N+1}; \end{pmatrix}$$
$$\times\, {}_{2N}\Psi_{2N}\left[\begin{matrix} a_1 b_1/a,\ldots, a_1 b_{2N}/a; \\ qa_1/b_1,\ldots, qa_1/b_{2N}; \end{matrix}\ \frac{q^N a^N}{b_1 b_2\ldots b_{2N}}\right]+$$
$$+\mathrm{idem}(a_1; a_2, a_3,\ldots, a_{N+1}), \tag{8}$$

† Jackson (3) (2.1), (2.3).

‡ Sears (3) (7.3), (7.4), (7.5).

which expresses a well-poised ${}_{2N}\Psi_{2N}$ in terms of $N+1$ other series of the same type, and

$$\Pi\begin{pmatrix} qa/b_1,\ldots,qa/b_{2N-1},q/b_1,\ldots,q/b_{2N-1},\pm\sqrt{(qa)},\pm\sqrt{(q/a)},\sqrt{a},-\sqrt{a}, \\ q/\sqrt{a},-q/\sqrt{a}; \\ a,q,a_1,\ldots,a_N,q/a,q/a_1,\ldots,q/a_N,a_1/a,\ldots,a_N/a,aq/a_1,\ldots,aq/a_N; \end{pmatrix}\times$$

$$\times\,{}_{2N-1}\Psi_{2N-1}\begin{bmatrix} b_1,b_2,\ldots,b_{2N-1}; & \mp q^{N-\frac{1}{2}}a^{N-\frac{1}{2}} \\ aq/b_1,aq/b_2,\ldots,aq/b_{2N-1}; & \overline{b_1 b_2\ldots b_{2N-1}} \end{bmatrix}$$

$$= a_1\,\Pi\begin{pmatrix} qa_1/b_1,\ldots,qa_1/b_{2N-1},qa/a_1b_1,\ldots,qa/a_1b_{2N-1}, \\ \pm a_1\sqrt{(q/a)},\pm\sqrt{(qa)}/a_1,\sqrt{a}/a_1,-\sqrt{a}/a_1,qa_1/\sqrt{a},-qa_1/\sqrt{a}; \\ a/a_1,q/a_1,aq/a_1^2,aq/a_1a_2,\ldots,aq/a_1a_N,a_2/a_1,\ldots,a_N/a_1, \\ a_1,a_1q/a,a_1^2/a,a_1a_2/a,\ldots,a_1a_N/a,q,a_1q/a_2,\ldots,a_1q/a_N; \end{pmatrix}\times$$

$$\times\,{}_{2N-1}\Psi_{2N-1}\begin{bmatrix} a_1b_1/a,a_1b_2/a,\ldots,a_1b_{2N-1}/a; & \mp q^{N-\frac{1}{2}}a^{N-\frac{1}{2}} \\ qa_1/b_1\,qa_1/b_2,\ldots,qa_1/b_{2N-1}; & \overline{b_1 b_2\ldots b_{2N-1}} \end{bmatrix}+$$

$$+\text{idem}(a_1;a_2,a_3,\ldots,a_N). \tag{9}$$

Here either the upper or the lower sign must be taken throughout. This expresses a well-poised ${}_{2N-1}\Psi_{2N-1}$ series in terms of N other series of the same type.

In these results, we can again reverse any of the Ψ series, and, by a suitable choice of parameters, we can reduce any of the Ψ series to Φ series, and so obtain restatements of Sears's results.

3. Next, I give the corresponding results for hypergeometric bilateral series, and I give a short proof of the main result, based on integrals of the Barnes type. With the usual notation for ordinary hypergeometric series, let

$$(a)_n\equiv a(a+1)\ldots(a+n-1),\quad (a)_0\equiv 1,\quad\text{and}\quad (a)_{-n}\equiv(-1)^n/(1-a)_n,$$

$${}_MF_{M-1}\begin{bmatrix} a_1,a_2,\ldots,a_M; & x \\ b_1,b_2,\ldots,b_{M-1}; & \end{bmatrix}\equiv\sum_{n=0}^{\infty}\frac{(a_1)_n(a_2)_n\ldots(a_M)_n\,x^n}{(b_1)_n(b_2)_n\ldots(b_{M-1})_n\,n!},$$

$${}_MH_M\begin{bmatrix} a_1,a_2,\ldots,a_M; & x \\ b_1,b_2,\ldots,b_M; & \end{bmatrix}\equiv\sum_{n=-\infty}^{\infty}\frac{(a_1)_n(a_2)_n\ldots(a_M)_n\,x^n}{(b_1)_n(b_2)_n\ldots(b_M)_n},$$

where $|x|=1$, for convergence, and

$$\Gamma\begin{pmatrix} a_1,a_2,\ldots,a_M; \\ b_1,b_2,\ldots,b_N; \end{pmatrix}\equiv\frac{\Gamma(a_1)\Gamma(a_2)\ldots\Gamma(a_M)}{\Gamma(b_1)\Gamma(b_2)\ldots\Gamma(b_N)}.$$

In (4), write $q^{b_1},\ldots,q^{b_M}$ for $b_1,\ldots,b_M$, $q^{c_1},\ldots,q^{c_M}$ for $c_1,\ldots,c_M$, $q^{a_1},\ldots,q^{a_M}$, for $a_1,\ldots,a_M$, and q^x for x. Let $x=a_1+a_2+\ldots+a_M-c_1-c_2-\ldots-c_M$,

and let $q \to 1$. Then $(aq/b)_{q,n}/(aq/c)_{q,n}$ becomes $(1+a-b)_n/(1+a-c)_n$ and in effect,

$$\Pi\begin{pmatrix} aq/b; \\ aq/c; \end{pmatrix} \quad \text{becomes} \quad \Gamma\begin{pmatrix} 1+a-c; \\ 1+a-b; \end{pmatrix}$$

for any integer n, positive or negative. The first series in (4) vanishes, and we have

$$\Gamma\begin{pmatrix} 1+a_2-a_1,\ldots, 1+a_M-a_1, a_1-a_2,\ldots, a_1-a_M; \\ 1+b_1-a_1,\ldots, 1+b_M-a_1, a_1-c_1,\ldots, a_1-c_M; \end{pmatrix}\times$$

$$\times {}_MH_M\begin{bmatrix} 1+c_1-a_1,\ldots, 1+c_M-a_1; & 1 \\ 1+b_1-a_1,\ldots, 1+b_M-a_1; & \end{bmatrix} + \text{idem}(a_1; a_2, a_3,\ldots, a_M) = 0. \quad (10)$$

This simple result is a general relation between M series of the type ${}_MH_M(1)$.

4. Next we consider the well-poised results. From (7), when $q \to 1$, we have

$$\Gamma\begin{pmatrix} a, a_1,\ldots, a_N, 1+a-a_1,\ldots, 1+a-a_N, 1-a, 1-a_1,\ldots, 1-a_N, \\ a_1-a,\ldots, a_N-a; \\ 1+a-b_1,\ldots, 1+a-b_{2N}, 1-b_1,\ldots, 1-b_{2N}, \tfrac{1}{2}a, 1-\tfrac{1}{2}a; \end{pmatrix}\times$$

$$\times {}_{2N}H_{2N}\begin{bmatrix} b_1,\ldots, b_{2N}; & \\ 1+a-b_1,\ldots, 1+a-b_{2N}; & -1 \end{bmatrix}$$

$$= \Gamma\begin{bmatrix} a-a_1, 1-a_1, a_1, a_2-a_1,\ldots, a_N-a_1, 2a_1-a, 1+a-2a_1, \\ 1+a_1-a, a_1+a_2-a,\ldots, a_1+a_N-a, 1+a_1-a_2,\ldots, 1+a_1-a_N, \\ 1+a-a_1-a_2,\ldots, 1+a-a_1-a_N; \\ 1+a_1-b_1,\ldots, 1+a_1-b_{2N}, 1+a-a_1-b_1,\ldots, 1+a-a_1-b_{2N}, \\ \tfrac{1}{2}a-a_1, 1+a_1-\tfrac{1}{2}a; \end{bmatrix}\times$$

$$\times {}_{2N}H_{2N}\begin{bmatrix} a_1+b_1-a,\ldots, a_1+b_{2N}-a; & -1 \\ 1+a_1-b_1,\ldots, 1+a_1-b_{2N}; & \end{bmatrix} + \text{idem}(a_1; a_2,\ldots, a_N). \quad (11)$$

From (8), letting $q \to 1$, we find

$$\Gamma\begin{pmatrix} a_1,\ldots, a_{N+1}, 1+a-a_1,\ldots, 1+a-a_{N+1}, 1-a_1,\ldots, 1-a_{N+1}, \\ a_1-a,\ldots, a_{N+1}-a; \\ 1+a-b_1,\ldots, 1+a-b_{2N}, 1-b_1,\ldots, 1-b_{2N}; \end{pmatrix}\times$$

$$\times {}_{2N}H_{2N}\begin{bmatrix} b_1,\ldots, b_{2N}; & 1 \\ 1+a-b_1,\ldots, 1+a-b_{2N}; & \end{bmatrix}$$

$$= \Gamma\begin{pmatrix} 1-a_1, 1+a-a_1, 1+a-a_1-a_2,\ldots, 1+a-a_1-a_{N+1}, \\ a_2-a_1,\ldots, a_{N+1}-a_1, a_1, a_1-a, a_1+a_2-a,\ldots, a_1+a_{N+1}-a, \\ 1+a_1-a_2,\ldots, 1+a_1-a_{N+1}; \\ 1+a_1-b_1,\ldots, 1+a_1-b_{2N}, 1+a-a_1-b_1,\ldots, 1+a-a_1-b_{2N}; \end{pmatrix}\times$$

$$\times {}_{2N}H_{2N}\begin{bmatrix} a_1+b_1-a,\ldots, a_1+b_{2N}-a; & 1 \\ 1+a_1-b_1,\ldots, 1+a_1-b_{2N}; & \end{bmatrix} + \text{idem}(a_1; a_2,\ldots, a_{N+1}). \quad (12)$$

Finally, letting $q \to 1$, in (9), we have

$$\Gamma\begin{pmatrix} a, 1+a-a_1,..., 1+a-a_N, a_1,..., a_N, 1-a, 1-a_1,..., 1-a_N, \\ a_1-a,..., a_N-a; \\ 1+a-b_1,..., 1+a-b_{2N-1}, 1-b_1,..., 1-b_{2N-1}, \tfrac{1}{2}a, 1-\tfrac{1}{2}a, \\ \tfrac{1}{4}(1-a)\mp\tfrac{1}{4}(1-a), \tfrac{1}{4}(1+a)\mp\tfrac{1}{4}(1+a); \end{pmatrix}\times$$

$$\times {}_{2N-1}H_{2N-1}\begin{bmatrix} b_1,..., b_{2N-1}; \\ 1+a-b_1,..., 1+a-b_{2N-1}; \end{bmatrix}\pm 1$$

$$= \Gamma\begin{bmatrix} a-a_1, 1-a_1, 1+a-2a_1, 1+a-a_1-a_2,..., 1+a-a_1-a_N, \\ a_2-a_1,..., a_N-a_1, a_1, 1+a_1-a, 2a_1-a, a_1+a_2-a,..., \\ a_1+a_N-a, 1+a_1-a_2,..., 1+a_1-a_N; \\ 1+a_1-b_1,..., 1+a_1-b_{2N-1}, 1+a-a_1-b_1,..., \\ 1+a-a_1-b_{2N-1}, \tfrac{1}{2}a-a_1, 1+a_1-\tfrac{1}{2}a, \\ \tfrac{1}{2}a_1+\tfrac{1}{4}(1-a)\mp\{\tfrac{1}{2}a_1+\tfrac{1}{4}(1-a)\}, -\tfrac{1}{2}a_1+\tfrac{1}{4}(1-a)\mp\{\tfrac{1}{4}(1-a)-\tfrac{1}{2}a_1\}; \end{bmatrix}\times$$

$$\times {}_{2N-1}H_{2N-1}\begin{bmatrix} a_1+b_1-a,..., a_1+b_{2N-1}-a; \\ 1+a_1-b_1,..., 1+a_1-b_{2N-1}; \end{bmatrix}\pm 1 +$$

$$+\mathrm{idem}(a_1; a_2,..., a_N), \quad (13) \text{ and } (14)$$

where the upper signs are to be taken throughout to give (13) and the lower signs throughout to give (14).

It has been pointed out to me by Professor Bailey that (14) follows from (11) by taking $b_{2N} = \tfrac{1}{2}(1+a)$, (11) follows from (13) by putting $N+1$ for N in (13) and then letting $b_{2N+1} \to -\infty$, $a_{N+1} \to \infty$, and (13) follows from (12) by taking $a_{N+1} = b_{2N} = \tfrac{1}{2}(1+a)$.

When $M = 2$, (10) gives a relation between two ${}_2H_2(1)$ series. If, further, $a_2 = b_2$, the ${}_2H_2(1)$ series on the right of the result reduces to a ${}_2F_1(1)$ series, which can then be summed by Gauss's theorem, and the sum of the series ${}_2H_2(1)$† can be deduced. When $M = 3$, and $a_1 = 1$, $a_2 = b_2$, $a_3 = b_3$, putting a for c_1, b for c_2, c for c_3, d for b_1, e for b_2, and f for b_3, we have

$${}_3H_3\begin{bmatrix} a, b, c; \\ d, e, f; \end{bmatrix} 1$$

$$= \Gamma\begin{pmatrix} d, e, e-f, 1-a, 1-b, 1-c; \\ 1-f, 1+d-e, e-a, e-b, e-c; \end{pmatrix} {}_3F_2\begin{bmatrix} 1+a-e, 1+b-e, 1+c-e; \\ 1+d-e, 1+f-e; \end{bmatrix} +$$

$$+\mathrm{idem}(e; f). \quad (15)$$

From (12), when $N = 2$, $a_1 = b_1$ and $a_2 = b_2$, the two series on the right reduce to well-poised ${}_3F_2(1)$ series which can be summed by Dixon's theorem‡ to give the sum§ of a well-poised ${}_3H_3(1)$ series. When $N = 3$, $a_1 = b_1$, $a_2 = b_2$, $a_3 = b_3 = 1+\tfrac{1}{2}a$, one of the ${}_5H_5(1)$ series vanishes, and

† Bailey (1) (1.3). ‡ Bailey (2) (3.1) (1). § Bailey (1) (2.5).

the other two on the right reduce to well-poised ${}_5F_4(1)$ series which can be summed to give, on reduction, the sum† of a well-poised ${}_5H_5(1)$. When $N = 4$, we have a relation between five well-poised ${}_7H_7(1)$ series, which becomes, on reduction, a relation expressing a well-poised ${}_7H_7(1)$ series with the special form of the first parameter in terms of three well-poised ${}_7F_6(1)$ series all with the special form of second parameters.

5. An extension of the result (10) can easily be proved by considering the integral

$$I \equiv \frac{1}{2\pi i}\int\limits_C \Gamma\begin{pmatrix} a_1+s,\dots, a_M+s, 1-a_1-s,\dots, 1-a_M-s; \\ b_1+s,\dots, b_M+s, 1-c_1-s,\dots, 1-c_M-s; \end{pmatrix} x^s\, ds$$

taken round a large circle C, on which $s = Re^{i\theta}$. On C the integrand is $O(s^{\mathrm{re}(c_1+\dots+c_M-b_1-\dots-b_M)})$, and so $I \to 0$ as $R \to \infty$ if

$$\mathrm{re}(b_1+\dots+b_M-c_1-\dots-c_M) > 0, \qquad |x| = 1.$$

But the integrand has poles at $s = -a_1-n,\dots, -a_M-n$, and at $s = 1-a_1+n,\dots, 1-a_M+n$, for $n = 0, 1, 2,\dots$. Hence from the residues we have

$$x^{-a_1}\Gamma\begin{pmatrix} 1+a_2-a_1,\dots, 1+a_M-a_1, a_1-a_2,\dots, a_1-a_M; \\ 1+b_1-a_1,\dots, 1+b_M-a_1, a_1-c_1,\dots, a_1-c_M; \end{pmatrix}\times$$
$$\times {}_MH_M\begin{bmatrix} 1+c_1-a_1,\dots, 1+c_M-a_1; \\ 1+b_1-a_1,\dots, 1+b_M-a_1; \end{bmatrix} x\Bigg] +$$
$$+\mathrm{idem}(a_1; a_2, a_3,\dots, a_M) = 0 \qquad (16)$$

where $|x| = 1$.

In a future paper I intend to discuss further Barnes's integrals and their basic analogues, which give simple proofs, not only of all the preceding results, but also of Sears's original theorems.

REFERENCES

1. W. N. Bailey, 'Series of hypergeometric type which are infinite in both directions', *Quart. J. of Math.* (Oxford) 7 (1936), 105–15.
2. —— *Generalized hypergeometric series* (Cambridge Tract, 1935).
3. M. Jackson, 'On well-poised bilateral hypergeometric series of the type ${}_8\Psi_8$', *Quart. J. of Math.* (Oxford), (2) 1 (1950), 63–68.
4. D. B. Sears, 'Transformation theory of basic hypergeometric functions of any order', *Proc. London Math. Soc.* (2) 53 (1951), 158–91.

† Bailey (1) (3.4).

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